Morita's duality for split reductive groups

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Abstract. In this paper, we extend the work in *Morita's Theory for the Symplectic Groups* [7] to split reductive groups. We construct and study the holomorphic discrete series representation and the principal series representation of a split reductive group G over a p-adic field F as well as a duality between certain sub-representations of these two representations.

Contents

Notations

Let p be a prime, F a finite extension of $\mathbb{Q}_p$, $\mathfrak{o}$ the ring of integers of F, ϖ a uniformizer of $\mathfrak{o}$, $|\ |$ the normalized absolute value, and F^{alg} an algebraic closure of F. Let K be an extension of F with an absolute value extending $|\ |$, and $\mathfrak{o}_K$ the valuation ring of K. We assume that K is complete with respect to $|\ |$, and moreover, K is spherically complete whenever topological properties of the K-vector spaces are under consideration.

1. Introduction

In a series of papers, [4], [5] and [6], Morita and Murase innitiated the work on the representation theory for $\mathrm{SL}(2, F)$ with coefficient field K, especially holomorphic discrete

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series representations and their duality relations with principal series representations. Principal series representations, or more generally, induced representations, appeared in many literatures, notably in Féaux de Lacroix's work [2] on locally analytic representations. On the other hand, holomorphic discrete series representations were not extensively studied. The holomorphic discrete series representation of $\mathrm{SL}(n+1, F)$ associated to a rational representation of $\mathrm{GL}(n, F)$ were introduced by Schneider in [8], in order to understand the de Rham complex over Drinfel'd's space as representation of $\mathrm{SL}(n+1, F)$. In another direction, the holomorphic discrete series representation of $\mathrm{Sp}(2n, F)$ associated to a K-rational representation of $\mathrm{GL}(n, F)$ were constructed in our recent work [7]. Furthermore, the algebraization and generalization of Morita's duality were established in [7].

The purpose of this paper is to generalize Morita's theory from $\mathrm{Sp}(2n, F)$ to a split reductive group G. We are able to do such a generalization owning to the entirely algebraic construction of Morita's theory for $\mathrm{Sp}(2n, F)$. Therefore, we shall closely follow the main ideas presented in [7].

In the first paragraph, we recollect some notions on split reductive groups and construct an F-regular function f on G that characterizes the parabolic big cell associated to a parabolic subgroup. In particular, f corresponds to the determinant function on $\mathrm{M}(n, F)$ used in the definition of p-adic Siegel upper half-space in [7] (see Example 2.5 and 4.3). f will appear extensively in the study of the rigid symmetric space associated to G and holomorphic series representations.

The principal series representation $(C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V), T_{\sigma})$ is another interpretation of the parabolic induction from a locally analytic K-representation (σ, V) of the Levi subgroup (cf. [5] and [7]). In the second paragraph, applying the general results of Féaux de Lacroix on induced representations of F-Lie groups ([2]), one sees that $(C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V), T_{\sigma})$ is a locally analytic representation of G over a K-vector space of compact type.

The third paragraph is dedicated to the construction and study of the rigid analytic symmetric space $\mathbf{\Omega}$, which is the foundation of holomorphic discrete series representations. Some examples are the p-adic upper half-plane for $\mathrm{SL}(2, F)$ (cf. [4]), Drinfel'd's space for $\mathrm{SL}(n+1, F)$ (cf. [8]) and the p-adic Siegel upper half-space for $\mathrm{Sp}(2n, F)$ (cf. [7]). Such symmetric spaces have been studied by van der Put and Voskuil in [14]. We shall however use another approach following [10] and [7] to construct the admissible affinoid covering, which enables us to obtain precise descriptions of rigid analytic functions on $\mathbf{\Omega}$. From this, we prove that the space $\mathscr{O}_K(\mathbf{\Omega})$ of K-rigid analytic functions on $\mathbf{\Omega}_K$ is a nuclear K-Fréchet space.

In the fourth paragraph, for a K-rational representation (σ, V) of the Levi subgroup, we construct the holomorphic discrete series representation $(\mathscr{O}_{\sigma}(\Omega), \pi_{\sigma})$ defined over the nuclear K-Fréchet space of V-valued rigid analytic functions on $\mathbf{\Omega}$. Moreover, we prove that its dual representation is locally analytic.

Since the strong duality gives rise to a contra-variant equivalence between the category of K-vector spaces of compact type and the category of nuclear K-Fréchet spaces (cf.

[11]), it is natural to expect certain duality relations between sub-quotient spaces of principal series representations and those of holomorphic discrete series representations. For $\mathrm{SL}(2, F)$, a duality of this kind via residues is analytically constructed by Morita (cf. [5]). However, there does not seem to be any direct way to generalize Morita's duality. Nevertheless, Morita's duality may be algebraically interpreted in a weaker form and generalized to any split reductive group G. These are done in the last paragraph. For a K-rational representation (σ, V) of the Levi subgroup, a closed sub-representation $B_{\sigma^*}(\mathfrak{H}, V^*)$ of $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)$ and a closed sub-representation $\mathscr{N}_\sigma(\Omega)$ of $\mathscr{O}_\sigma(\Omega)$ are algebraically constructed, along with a duality between them. As discussed in [7], our duality for $\mathrm{SL}(2, F)$ is exactly Morita's duality composed with Casselman's intertwining operator (cf. [5]).

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2. A lemma on the split reductive groups

2.1. Split reductive groups. We adopt the notations in [3] Part II, Chapter 1.

Let G be a connected split reductive algebraic group over F, T a split maximal torus of G. We have the decomposition of Lie algebra $\mathfrak{g}$ of G (over F) in the form

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \tag{2.1}$$

where $\mathfrak{g}_0$ is the Lie algebra of T and R is the root system of G with respect to T.

Each $\mathfrak{g}_\alpha$ is of rank 1 over F, and we denote $\mathrm{U}_\alpha \simeq \mathbb{G}_a(F)$ the root subgroup of G corresponding to α.

Let $W \cong N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ be the Weyl group of R. For $w \in W$, we also denote w a representing element in $N_{\mathrm{G}}(\mathrm{T})$.

Choose a positive system R^+ and denote S the corresponding set of simple roots. Let B^+ denote the corresponding Borel subgroup and B^- its opposite Borel subgroup, $\mathrm{U}^\pm_{\mathrm{G}} = \mathrm{U}(\pm R^+)$ the unipotent radical of $\mathrm{B}^\pm$.

Throughout this article we fix a subset I of S, and denote $R_I = R \cap \mathbb{Z}I$, W_I the Weyl group of R_I, $R^+_I = R^+ - R_I$, P^+ the standard parabolic subgroup of G corresponding to R^+_I, P^- its opposite parabolic subgroup, $\mathrm{U}^\pm = \mathrm{U}(\pm R^+_I)$ the unipotent radical of $\mathrm{P}^\pm$, L the common Levi subgroup of P^+ and P^-.

L is a split reductive group with split maximal torus T, root system R_I, positive system $R^+_{\mathrm{L}} = R_I \cap R^+$ and Weyl group W_I. Let $\mathrm{U}^\pm_{\mathrm{L}} = \mathrm{U}(\pm R^+_{\mathrm{L}})$.

We recall ([3] Part II.1.7) that for any closed and unipotent subset R' of R (that is, $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap R \subset R'$ for any $\alpha, \beta \in R'$ and $R' \cap (-R') = \emptyset$), for instance $\pm R^+$, $\pm R^+_I$ and $\pm R^+_{\mathrm{L}}$, the multiplication induces, for any ordering of R', an isomorphism of schemes over F

$$\prod_{\alpha \in R'} \mathrm{U}_\alpha \xrightarrow{\simeq} \mathrm{U}(R'). \tag{2.2}$$

2.2. The parabolic big cell. We have the Bruhat decomposition of G ([3] Part II.1.9)

$$\mathrm{G} = \bigcup_{w\in W} \mathrm{B}^- w \mathrm{B}^+ = \bigcup_{w\in W} \mathrm{U}_{\mathrm{G}}^- w \mathrm{T} \mathrm{U}_{\mathrm{G}}^+.$$

Let C denote the parabolic big cell

$$\mathrm{C} = \bigcup_{w\in W_I} \mathrm{U}_{\mathrm{G}}^- w \mathrm{T} \mathrm{U}_{\mathrm{G}}^+ = \mathrm{U}^-\Big(\bigcup_{w\in W_I} \mathrm{U}_{\mathrm{L}}^- w \mathrm{T} \mathrm{U}_{\mathrm{L}}^+\Big)\mathrm{U}^+ = \mathrm{U}^-\mathrm{L}\mathrm{U}^+ = \mathrm{U}^-\mathrm{P}^+ = \mathrm{P}^-\mathrm{U}^+.$$

Then G is the disjoint union of C and $\mathrm{U}_{\mathrm{G}}^- w \mathrm{T} \mathrm{U}_{\mathrm{G}}^+$ for all $w \notin W_I$.

Let $r = |R_I^+| = \dim \mathrm{U}^+$. We consider the adjoint representation of G on $\bigwedge^r \mathfrak{g}$ over F. From the decomposition (2.1) of $\mathfrak{g}$ we obtain a direct sum decomposition of $\bigwedge^r \mathfrak{g}$. Choosing X_α a nonzero element in $\mathfrak{g}_\alpha$ for each $\alpha \in R$ and a basis of $\mathfrak{g}_0$ we obtain a basis of $\bigwedge^r \mathfrak{g}$ with respect to this decomposition and containing $Y = \bigwedge_{\alpha\in R_I^+} X_\alpha$. For $g \in \mathrm{G}$ we define $f(g)$ to be the coefficient of Y in the expansion of $\mathrm{Ad}(g)Y$ in the chosen basis of $\bigwedge^r \mathfrak{g}$. Then f is a regular function on G over F.

There is a partial order on $\mathbb{Z}S$: $\gamma \prec \delta$ iff $\delta - \gamma$ is a sum of positive roots. If one considers the group action of the symmetric group S_r on $(\mathbb{Z}S)^r$ via coordinate permutation, the set of the unordered r-tuples $[\gamma_1, ..., \gamma_r]$ of elements in $\mathbb{Z}S$ may be viewed as the set of S_r-orbits in $(\mathbb{Z}S)^r$. We define $[\gamma_1, ..., \gamma_r] \prec [\delta_1, ..., \delta_r]$ iff there exists $s \in S_r$ such that $\gamma_j \preceq \delta_{s(j)}$ for all $1 \leqslant j \leqslant r$ and $\gamma_j \prec \delta_{s(j)}$ for at least one j.

We adopt the convention that $\mathfrak{g}_\gamma = 0$ if $\gamma \in \mathbb{Z}S$ is nonzero and not a root. Then $[\mathfrak{g}_\beta, \mathfrak{g}_\alpha] \subset \mathfrak{g}_{\alpha+\beta}$, and therefore

$$\mathrm{Ad}(u_\beta)X_\alpha \in X_\alpha + \sum_{i\geqslant 1} \mathfrak{g}_{\alpha+i\beta}, \quad u_\beta \in \mathrm{U}_\beta.$$

If $\alpha, \beta \in R_I^+$, then it is clear that either $\alpha + i\beta \in R_I^+$ or $\mathfrak{g}_{\alpha+i\beta} = 0$. The same statement holds for $\alpha \in R_I^+$ and $\beta \in R_I$, since the β-string through α lies in R_I^+. Therefore U_β fixes Y for any $\beta \in R_I^+ \cup R_I = R^+ \cup (-R_I^+)$. (2.2) implies that

(2.3) $\quad Y$ is invariant under $\mathrm{U}_{\mathrm{G}}^+$ and $\mathrm{U}_{\mathrm{L}}^-$.

If we let β be negative roots, then (2.2) also implies that for $v^- \in \mathrm{U}_{\mathrm{G}}^-$,

$$\mathrm{Ad}(v^-)X_\alpha \in X_\alpha + \sum_{\gamma\prec\alpha} \mathfrak{g}_\gamma. \tag{2.4}$$

For $t \in T$,

$$\mathrm{Ad}(t)Y = \prod_{\alpha\in R_I^+} \alpha(t)Y. \tag{2.5}$$

For $w \in W$ there exists a constant $c_{w,\alpha} \in F^\times$ satisfying

$$\mathrm{Ad}(w)X_\alpha = c_{w,\alpha}X_{w\alpha}. \tag{2.6}$$

In view of (2.2), we see that w preserves R_I^+ iff w normalizes U^+. Since $N_{\mathrm{G}}(\mathrm{U}^+) = \mathrm{P}^+$ and $w \in \mathrm{P}^+$ iff $w \in W_I$,

(2.7) $\quad w$ preserves R_I^+ iff $w \in W_I$.

Since $\mathrm{P}^+ = \bigcup_{w\in W_I} \mathrm{U}_\mathrm{L}^- w\mathrm{T}\mathrm{U}_\mathrm{G}^+$, if we write $p^+ = u^- wtv^+$ ($v^+ \in \mathrm{U}_\mathrm{G}^+$, $t \in \mathrm{T}$, $w \in W_I$, $u^- \in \mathrm{U}_\mathrm{L}^-$), then it follows from (2.3), (2.5), (2.6) and (2.7) that

$$\mathrm{Ad}(p^+)Y = \mathrm{sign}(w) \prod_{\alpha\in R_I^+} c_{w,\alpha}\alpha(t) \cdot Y,$$

where $\mathrm{sign}(w)$ denotes the sign of the permutation w on R_I^+. Moreover, it follows from (2.4) that

$$\mathrm{Ad}(v^- wtv^+)Y \in \ \mathrm{sign}(w) \prod_{\alpha\in R_I^+} c_{w,\alpha}\alpha(t) \cdot Y + \sum_{[\gamma_j]<[\alpha]_{\alpha\in R_I^+}} \bigwedge_{j=1}^{r} \mathfrak{g}_{\gamma_j}.$$

So

$$f(v^- wtv^+) = \mathrm{sign}(w) \prod_{\alpha\in R_I^+} c_{w,\alpha}\alpha(t).$$

Similarly, for $w \notin W_I$,

$$\mathrm{Ad}(v^- wtv^+)Y \in \sum_{[\gamma_j]\leq[w\alpha]_{\alpha\in R_I^+}} \bigwedge_{j=1}^{r} \mathfrak{g}_{\gamma_j}.$$

It follows from the proof of (2.3) that $\alpha + \beta \in R_I^+$ or $\mathfrak{g}_{\alpha+\beta} = 0$ if $\alpha \in R_I^+$ and $\beta \in R^+$, so $\alpha \in R_I^+$ and $\delta \geq \alpha$ imply $\delta \in R_I^+$ or $\mathfrak{g}_\delta = 0$. Therefore if $\delta \in \{0\} \cup R - R_I^+$ and $\gamma \leq \delta$ then $\gamma \notin R_I^+$, and hence (2.7) implies that Y does not appear in the expression of $\mathrm{Ad}(v^- wtv^+)Y$, so $f(v^- wtv^+) = 0$.

We conclude with the following lemma.

Lemma 2.1. *Let the notations be as above, then*

(1) *For* $p^+ \in \mathrm{P}^+$,

$$\mathrm{Ad}(p^+)Y = f(p^+)Y,$$

and hence f *is an* F*-rational character on* P^+.

(2) *For* $w \in W_I$,

$$f(v^- wtv^+) = \mathrm{sign}(w) \prod_{\alpha\in R_I^+} c_{w,\alpha}\alpha(t), \quad v^\pm \in \mathrm{U}_\mathrm{G}^\pm,\ t \in \mathrm{T}.$$

(3) f *vanishes on* $\mathrm{U}_\mathrm{G}^- w\mathrm{T}\mathrm{U}_\mathrm{G}^+$ *for* $w \notin W_I$.

In particular,

(4) C *is an open* F*-subscheme of* G*, and* $F[\mathrm{C}] = F[\mathrm{G}]_f$.

(5) f *is right invariant under* U_G^+ *and left invariant under* U_G^-.

Example 2.2. *[cf. [3] Part II 1.9 and [13] §5 Theorem 7] If* $I = \emptyset$*, then* $\mathrm{L} = \mathrm{T}$ *and* $\mathrm{U}^\pm = \mathrm{U}_\mathrm{G}^\pm$*. Lemma 2.1 implies that* $f(u^- tu^+) = \prod_{\alpha\in R^+} \alpha(t)$ *and* $f(u^- wtu^+) = 0$ *for all nontrivial* w*,* $u^\pm \in \mathrm{U}_\mathrm{G}^\pm$ *and* $t \in \mathrm{T}$.

Example 2.3. *Let* $\mathrm{G} = \mathrm{SL}(n+1, F)$. *Write* $g = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$ *with* $A \in \mathrm{M}(n, F), B \in \mathrm{M}(n, 1; F), C \in \mathrm{M}(1, n; F)$ *and* $d \in F$. *Let*

$$\mathrm{U}^+ = \left\{ \begin{pmatrix} I_n & 0 \\ C & 1 \end{pmatrix} \in \mathrm{G} \right\},$$

$$\mathrm{L} = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in \mathrm{G} \right\}.$$

Some calculations show that

$$f(g) = d^{n+1}.$$

Example 2.4. *More generally, we consider* $\mathrm{G} = \mathrm{GL}(n, F)$. *Let* $(n_1, ..., n_s)$ *be a partition of n. Write* $g = (g_{ij})_{1 \leqslant i, j \leqslant s}$ *with* $g_{ij} \in \mathrm{M}(n_i, n_j; F)$. *Let* U^+ *be the subgroup consisting of the matrices* $u = (u_{ij})$ *such that* $u_{ij} = 0$ *for* $j < i$ *and* $u_{ii} = I_{n_i}$, *and* L *the subgroup consisting of the matrices* $l = (l_{ij})$ *such that* $l_{ij} = 0$ *for* $i \neq j$. $\mathrm{L} \cong \prod_{1 \leqslant i \leqslant s} \mathrm{GL}(n_i, F)$.

For $l \in \mathrm{L}$,

$$\begin{aligned} f(l) &= \prod_{1 \leqslant j < i \leqslant s} \det(l_{ii})^{n_j} \det(l_{jj})^{-n_i} \\ &= \prod_{1 \leqslant i \leqslant s} \det(l_{ii})^{\sum_{j<i} n_j - \sum_{i<k} n_k}. \end{aligned}$$

The computation of the explicit formula for f involves the process of block lower (or upper) triangularization, and it turns out to be complicated if $s \geqslant 3$. *For* $s = 2$, *we have*

$$f(g) = \det(g_{22})^{n_1+n_2} \det(g)^{-n_2}.$$

The situation is the same if $\mathrm{G} = \mathrm{SL}(n, F)$. *For instance, if* $s = 2$, *then*

$$f(g) = \det(g_{22})^{n_1+n_2}.$$

Example 2.5. *We consider*

$$\mathrm{G} = \mathrm{Sp}(2n, F) = \left\{ g \in \mathrm{GL}(2n, F) \ :\ {}^t g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with* $A, B, C, D \in \mathrm{M}(n, F)$. *Let*

$$\mathrm{U}^+ = \left\{ \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} \ :\ C \in \mathrm{Sym}(n, F) \right\},$$

$$\mathrm{L} = \left\{ \begin{pmatrix} {}^t D^{-1} & 0 \\ 0 & D \end{pmatrix} \ :\ D \in \mathrm{GL}(n, F) \right\},$$

where $\mathrm{Sym}(n, F)$ *is the group of symmetric matrices of order n over* F. *Some calculations show that*

$$f(g) = \det(D)^{n+1}.$$

3. $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ and the principal series $(C^{\mathrm{an}}_\sigma(\mathfrak{H}, V), T_\sigma)$

Induced representations, especially the parabolic inductions, are of extreme importance in Lie theory. For p-adic Lie groups, they were studied by Féaux de Lacroix in his work ([2]) on the locally analytic representations.

We first recall the notion of locally analytic representations over K of an F-Lie group.

Definition 3.1 (cf. [2] and [11] §3). *A* locally analytic *representation* (σ, V) *of an F-Lie group* G *on a barreled locally convex Hausdorff K-vector space V is a continuous representation such that the orbit maps are V-valued locally analytic functions on* G. *More precisely, for any $v \in V$ there exists a BH-space W of V (that is, a Banach space W together with a continuous injection $W \hookrightarrow V$) such that $g \mapsto \sigma(g)v$ expands (in a neighborhood of the unit element) to a power series with W-coefficients.*

Let (σ, V) be a locally analytic representation of the Levi subgroup L. σ extends to a representation of P^- defined by $\sigma(ul) = \sigma(l)$ ($l \in \mathrm{L}$, $u \in \mathrm{U}^-$).

Definition 3.2. *Let* $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ *be the space of V-valued locally analytic functions ϕ on* G *satisfying*

$$\phi(p^- g) = \sigma(p^-)\phi(g), \quad \textit{for all } g \in \mathrm{G}, p^- \in \mathrm{P}^-.$$

On $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ *we have a* G*-action defined by right translation.*

Since the quotient space $\mathrm{P}^-\backslash\mathrm{G}$ is compact, we obtain from [2] 4.1.5 the following proposition.

Proposition 3.3. $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ *is a locally analytic representation of* G.

Next, we introduce the principal series representation, serving as another description of $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$.

Let $\mathfrak{H}$ and $\overline{\mathfrak{H}}$ denote the G-homogeneous spaces $\mathrm{U}^-\backslash\mathrm{G}$ and $\mathrm{P}^-\backslash\mathrm{G}$ respectively, and denote $\hat{g} := \mathrm{pr}^{\mathrm{G}}_{\mathfrak{H}}(g)$. Because $\mathrm{P}^- \cong \mathrm{U}^- \rtimes \mathrm{L}$, there is a left L-action on $\mathfrak{H}$, and $\overline{\mathfrak{H}} = \mathrm{L}\backslash\mathfrak{H}$.

Definition 3.4. *Let* $C^{\mathrm{an}}_\sigma(\mathfrak{H}, V)$ *be the space of V-valued locally analytic functions φ on $\mathfrak{H}$ satisfying*

$$\varphi(l\hat{g}) = \sigma(l)\varphi(\hat{g}), \quad \textit{for all } \hat{g} \in \mathfrak{H} \textit{ and } l \in \mathrm{L}.$$

The principal series representation $(C^{\mathrm{an}}_\sigma(\mathfrak{H}, V), T_\sigma)$ *of* G *is defined via*

$$(T_\sigma(g)\varphi)(\widehat{g'}) := \varphi(\widehat{g'} \cdot g).$$

Lemma 3.5.

(1) *The representation* $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ *is (naturally) isomorphic to* $(C^{\mathrm{an}}_\sigma(\mathfrak{H}, V), T_\sigma)$.

(2) $\mathrm{Ind}_{\mathrm{P}^-}^{\mathrm{G}}\sigma$ *is isomorphic to* $C^{\mathrm{an}}(\overline{\mathfrak{H}}, V)$.

(3) *Let ι be a locally analytic section of* $\mathrm{pr}^{\mathfrak{H}}_{\overline{\mathfrak{H}}}$, *then ι induces an isomorphism*

$$\begin{aligned} \iota^\circ : C^{\mathrm{an}}_\sigma(\mathfrak{H}, V) &\to C^{\mathrm{an}}(\overline{\mathfrak{H}}, V) \\ \varphi &\mapsto \varphi \circ \iota. \end{aligned}$$

Proof. (1) From a locally analytic section $\bar{\iota}$ of $\mathrm{pr}^{\mathrm{G}}_{\mathfrak{H}}$ we obtain an isomorphism ([2] 4.3.1)

$$\bar{\iota}^{\circ} : \mathrm{Ind}^{\mathrm{G}}_{\mathrm{U}^-}\mathbf{1} \simeq C^{\mathrm{an}}(\mathfrak{H}, V), \quad \phi \mapsto \phi \circ \bar{\iota}.$$

By restriction to the subspaces, $\bar{\iota}^{\circ}$ induces an isomorphism, independent of $\bar{\iota}$, between $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{P}^-}\sigma$ and $C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V)$. G-equivariance is evident.

(2) A locally analytic section $\tilde{\iota}$ of $\mathrm{pr}^{\mathrm{G}}_{\overline{\mathfrak{H}}}$ induces an isomorphism $\tilde{\iota}^{\circ}$ from $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{P}^-}\sigma$ onto $C^{\mathrm{an}}(\overline{\mathfrak{H}}, V)$ (ibid.).

(3) Choose $\bar{\iota}$ and $\tilde{\iota}$ compatible with ι, that is, $\tilde{\iota} = \bar{\iota} \circ \iota$, then the assertion follows from (1) and (2). Q.E.D.

Compactness of $\overline{\mathfrak{H}}$ implies that $C^{\mathrm{an}}(\overline{\mathfrak{H}}, V)$ is of compact type ([11] Lemma 2.1). By [11] Proposition 1.2, Theorem 1.3 and [9] Proposition 16.10, we have the following corollary.

Corollary 3.6. *Suppose that K is spherically complete. Let B be a closed subspace of $C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V)$, then both B and $C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V)/B$ are of compact type. In particular, they are reflexive, bornological, and complete. Moreover, their strong duals B^*_b and $(C^{\mathrm{an}}_{\sigma}(\mathfrak{H}, V)/B)^*_b$ are nuclear Fréchet spaces.*

4. Rigid analytic symmetric space $\mathbf{\Omega}$

The rigid analytic symmetric space $\mathbf{\Omega}$ associated to G (with respect to a parabolic P^+) was constructed by van der Put and Voskuil in [14]. Some examples are the p-adic upper half-plane, Drinfel'd's space and the p-adic Siegel upper half-space, which are associated to $\mathrm{SL}(2, F)$, $\mathrm{SL}(n+1, F)$ and $\mathrm{Sp}(2n, F)$ respectively (cf. [4], [10] and [7]).

4.1. Definition of the symmetric space $\mathbf{\Omega}$. Let $\mathbf{G}$, $\mathbf{P}^{\pm}$, $\mathbf{U}^{\pm}$, $\mathbf{L}$ and $\mathbf{C}$ denote the F-rigid analytifications of G, $\mathrm{P}^{\pm}$, $\mathrm{U}^{\pm}$, L and C respectively. f defines a rigid analytic function on $\mathbf{G}$.

Since f is left invariant under U^- (Lemma 2.1 (5)), we may define

$$f(\hat{g}, \mathbf{u}) := f(g \cdot \mathbf{u})$$

for $\hat{g} \in \mathfrak{H}$ and $\mathbf{u} \in \mathbf{U}^-$.

Definition 4.1. *Let*

$$\begin{aligned}\mathbf{\Omega} &:= \{\mathbf{u} \in \mathbf{U}^- \ : \ g \cdot \mathbf{u} \in \mathbf{C}, \textit{ for all } g \in \mathrm{G}\} \\ &= \{\mathbf{u} \in \mathbf{U}^- \ : \ f(\hat{g}, \mathbf{u}) \neq 0, \textit{ for all } \hat{g} \in \mathfrak{H}\}.\end{aligned}$$

We call $\mathbf{\Omega}$ the symmetric space associated to G *with respect to* P^+.

Example 4.2. *In the situation of Example 2.3, $\mathbf{U}^- \cong \mathbf{A}^n_{/F}$ and $(z_1, ..., z_n) \in \mathbf{\Omega}$ is given by the inequalities*

$$c_1 z_1 + ... + c_n z_n + d \neq 0 \quad \textit{for all nonzero } (c_1, ..., c_n, d) \in F^{n+1}.$$

Therefore $\mathbf{\Omega}$ is Drinfel'd's space.

EXAMPLE 4.3. *In the situation of Example 2.5,* $\mathrm{U}^- \cong \mathbf{Sym}(n)$ *and* $Z \in \mathbf{\Omega}$ *is given by the inequalities*

$$\det(CZ+D) \neq 0 \quad \textit{for all } C\,{}^tD = D\,{}^tC,\ \mathrm{rank}(C\ D) = n.$$

Therefore $\mathbf{\Omega}$ *is the* p*-adic Siegel upper half-space.*

We may also interpret $\mathbf{\Omega}$ to be the complement of all the G-translations of $(\mathbf{G}-\mathbf{C})/\mathbf{P}^+ = \mathbf{G}/\mathbf{P}^+ - \mathbf{U}^-$ in $\mathbf{G}/\mathbf{P}^+$. Therefore we have a left G-action on $\mathbf{\Omega}$ (induced from the left G-action on $\mathbf{G}/\mathbf{P}^+$). We denote $g * \mathbf{u}$ the action of $g \in \mathrm{G}$ on $\mathbf{u} \in \mathbf{\Omega}$. We have $g * \mathbf{u} = \mathrm{pr}^{\mathbf{C}}_{\mathbf{U}^-}(g \cdot \mathbf{u})$.

4.2. Automorphy factor. We define the *automorphy factor*

$$\begin{aligned} j : \mathrm{G} \times \mathbf{\Omega} &\to \mathbf{P}^+ \\ (g, \mathbf{u}) &\mapsto (g * \mathbf{u})^{-1} \cdot g \cdot \mathbf{u}. \end{aligned}$$

Then $j(g, \mathbf{u}) = \mathrm{pr}^{\mathbf{C}}_{\mathbf{P}^+}(g \cdot \mathbf{u})$, and straightforward computations show

$$j(g_1 g_2, \mathbf{u}) = j(g_1, g_2 * \mathbf{u}) j(g_2, \mathbf{u}). \tag{4.1}$$

For any $u \in \mathrm{U}^-$, $j(u, \mathbf{u}) = 1_{\mathbf{G}}$, and hence (4.1) implies $j(u \cdot g, \mathbf{u}) = j(g, \mathbf{u})$, so we may define $j(\hat{g}, \mathbf{u}) := j(g, \mathbf{u})$.

For $l \in \mathrm{L}$, since $l * \mathbf{u} = l \cdot \mathbf{u} \cdot l^{-1}$, we have $j(l, \mathbf{u}) = l$, and by (4.1)

$$j(l \cdot \hat{g}, \mathbf{u}) = l \cdot j(\hat{g}, \mathbf{u}). \tag{4.2}$$

Since f is left invariant under $\mathbf{U}^-$ (Lemma 2.1(5)), it follows that

$$f(j(\hat{g}, \mathbf{u})) = f(\hat{g}, \mathbf{u}). \tag{4.3}$$

From Lemma 2.1 (1), (4.1) and (4.3), we see that

$$f(\widehat{g_1 g_2}, \mathbf{u}) = f(\widehat{g_1}, g_2 * \mathbf{u}) f(\widehat{g_2}, \mathbf{u}). \tag{4.4}$$

Moreover, Lemma 2.1 (1), (4.2) and (4.3) imply

$$f(l \cdot \hat{g}, \mathbf{u}) = f(l) f(\hat{g}, \mathbf{u}). \tag{4.5}$$

4.3. The F-rigid analytic structure on $\mathbf{\Omega}$. van der Put and Voskuil defined an affinoid covering of $\mathbf{\Omega}$ using Bruhat-Tits Buildings ([14]). In this paper we choose another approach following the construction of affinoid covering of Drinfel'd's space in [10] and that of p-adic Siegel upper half-space in [7]. We endow $\mathbf{\Omega}$ with a structure of F-rigid analytic variety and show that it is an admissible open subset of $\mathbf{U}^-$ and, in particular, an open rigid analytic subspace of $\mathbf{U}^-$ (and therefore $\mathbf{G}/\mathbf{P}^+$).

We realize G as a Zariski closed subgroup of $\mathrm{GL}(n, F)$ such that T consists of diagonal matrices and B^+ consists of lower triangular matrices. Then $f(g)$ extends to an F-regular function on $\mathrm{GL}(n, F)$ with respect to the coordinates $g_{i,j}$ ($1 \leqslant i, j \leqslant n$), and $\det(g)^r f(g)$ is a homogeneous F-polynomial in $g_{i,j}$. We denote N the degree of $\det(g)^r f(g)$ and let M be an integer such that all the coefficients have absolute values bounded by $|\varpi|^{NM}$.

LEMMA 4.4. $\mathbf{\Omega}$ *is nonempty.*

Proof. It suffices to prove that G-translations of $\mathbf{G} - \mathbf{C}$ do not cover $\mathbf{G}$. For $g \in \mathrm{G}$, $g \cdot (\mathbf{G} - \mathbf{C})$ is the locus of $f(g^{-1} \cdot \mathbf{g}) = 0$. With the embedding of $\mathbf{G}$ into $\mathbf{GL}(n, F)$ we view $f(g^{-1} \cdot \mathbf{g})$ as a rational function in $\mathbf{g}_{ij}$ with $F[\mathrm{G}]$-coefficients, and, for a given $g \in \mathrm{G}$, $f(g^{-1} \cdot \mathbf{g})$ is a nonzero F-rational function in $\mathbf{g}_{ij}$. It is not hard to see that there are choices of $\mathbf{g}_{ij} \in F^{\mathrm{alg}}$ with appropriate absolute values so that the non-vanishing monomials in $f(g^{-1} \cdot \mathbf{g})$ are of distinct absolute values in $|(F^{\mathrm{alg}})^\times| = |\varpi|^{\mathbb{Q}}$ modulo $|F^\times| = |\varpi|^{\mathbb{Z}}$. Therefore there exists $\mathbf{g} \in \mathbf{G}$ such that $f(g^{-1} \cdot \mathbf{g})$ is nonzero for all $g \in \mathrm{G}$, and consequently $\mathbf{g}$ lies in the complement of all the $g \cdot (\mathbf{G} - \mathbf{C})$. Q.E.D.

Let $\mathrm{G}_{\mathfrak{o}}$ and $\mathrm{L}_{\mathfrak{o}}$ denote the intersections of G and L with $\mathrm{GL}(n, \mathfrak{o})$ respectively, and denote $\mathfrak{H}_{\mathfrak{o}} = \mathrm{pr}^{\mathrm{G}}_{\mathfrak{H}}(\mathrm{G}_{\mathfrak{o}})$.

We recall Iwasawa's decomposition

$$\mathrm{G} = \mathrm{B}^- \mathrm{G}_{\mathfrak{o}}.$$

Then $\mathrm{G} = \mathrm{P}^- \cdot \mathrm{G}_{\mathfrak{o}}$ and $\mathfrak{H} = \mathrm{L} \cdot \mathfrak{H}_{\mathfrak{o}}$, so (4.5) and Lemma 2.1 imply

$$\mathbf{\Omega} = \left\{ \mathbf{u} \in \mathbf{U}^- \ : \ f(\hat g, \mathbf{u}) \neq 0, \text{ for any } \hat g \in \mathfrak{H}_{\mathfrak{o}} \right\}.$$

For $\mathbf{u} \in \mathbf{U}^-$ an upper triangular matrix with diagonal entries 1, let

$$|\mathbf{u}| := \max_{1 \leqslant i \leqslant j \leqslant n} |\mathbf{u}_{ij}| = \max_{1 \leqslant i < j \leqslant n} \left\{ 1, |\mathbf{u}_{ij}| \right\}.$$

For any nonnegative integer m and $\hat g \in \mathfrak{H}_{\mathfrak{o}}$, we define

$$\mathbf{B}(m; \hat g) := \left\{ \mathbf{u} \in \mathbf{U}^- \ : \ |f(\hat g, \mathbf{u})| < |\mathbf{u}|^N \, |\varpi|^{N(M+m)} \right\}.$$

Lemma 4.5. *If m is a nonnegative integer and $g_1, g_2 \in \mathrm{G}_{\mathfrak{o}}$ such that $g_1 \equiv l \cdot g_2 \mod \varpi^{Nm+1}$ for some $l \in \mathrm{L}_{\mathfrak{o}}$, then*

$$\mathbf{B}(m; \widehat{g_1}) = \mathbf{B}(m; \widehat{g_2}).$$

Proof. Since $f|_{\mathrm{L}_{\mathfrak{o}}}$ is a continuous F-character (Lemma 2.1 (1)) and $\mathrm{L}_{\mathfrak{o}}$ is compact, the image of $\mathrm{L}_{\mathfrak{o}}$ under f is contained in $\mathfrak{o}^\times$. Therefore (4.5) implies $|f(\widehat{l g_2}, \mathbf{u})| = |f(\widehat{g_2}, \mathbf{u})|$, and hence $\mathbf{B}(m; \widehat{g_2}) = \mathbf{B}(m; \widehat{l g_2})$. So we may assume $g_1 \equiv g_2 \mod \varpi^{Nm+1}$.

We choose $\lambda \in (F^{\mathrm{alg}})^\times$ such that $|\lambda| = |\mathbf{u}|$. Since $|\lambda^{-1} \mathbf{u}_{ij}| \leqslant 1$,

$$g_1 \cdot \lambda^{-1} \mathbf{u} \equiv g_2 \cdot \lambda^{-1} \mathbf{u} \mod \varpi^{Nm+1},$$

and the matrices on both sides have entries with absolute values $\leqslant 1$. Applying $\det^r \cdot f$, we obtain

$$\lambda^{-N} \det(g_1)^r f(g_1 \cdot \mathbf{u}) \equiv \lambda^{-N} \det(g_2)^r f(g_2 \cdot \mathbf{u}) \mod \varpi^{NM+Nm+1},$$

and consequently

$$|\mathbf{u}|^{-N} \left| f(\widehat{g_1}, \mathbf{u}) \right| < |\varpi|^{N(M+m)} \Leftrightarrow |\mathbf{u}|^{-N} \left| f(\widehat{g_2}, \mathbf{u}) \right| < |\varpi|^{N(M+m)}.$$

Therefore $\mathbf{B}(m; \widehat{g_1}) = \mathbf{B}(m; \widehat{g_2})$. Q.E.D.

Let

$$\begin{aligned}\mathbf{\Omega}(m;\hat{g}) &:= \mathbf{U}^{-} - \mathbf{B}(m;\hat{g}) \\ &= \left\{\mathbf{u}\in\mathbf{U}^{-} \ :\ |f(\hat{g},\mathbf{u})| \geqslant |\mathbf{u}_{ij}|^{N}|\varpi|^{N(M+m)}, 1\leqslant i\leqslant j\leqslant n\right\}, \\ \mathbf{\Omega}(m) &:= \bigcap_{\hat{g}\in\mathfrak{H}_{\mathfrak{o}}} \mathbf{\Omega}(m;\hat{g}).\end{aligned}$$

For a given $\mathbf{u}\in\mathbf{\Omega}$, $|f(\hat{g},\mathbf{u})|$ has a positive lower bound on $\mathfrak{H}_{\mathfrak{o}}$. Therefore

$$\mathbf{\Omega} = \bigcup_{m=0}^{\infty}\mathbf{\Omega}(m).$$

Let $\mathfrak{H}^{(m)}$ be any finite subset of $\mathfrak{H}_{\mathfrak{o}}$ including a set of representatives in $\mathfrak{H}_{\mathfrak{o}}$ for $\mathrm{pr}^{\mathrm{G}}_{\mathfrak{H}}\left(\mathrm{L}_{\mathfrak{o}}\backslash\mathrm{G}_{\mathfrak{o}}/\mathrm{G}_{\mathfrak{o}}(Nm+1)\right)$, where $\mathrm{G}_{\mathfrak{o}}(Nm+1)$ denotes the congruence subgroup $(I_n+\varpi^{Nm+1}\mathrm{M}(n,\mathfrak{o}))\cap\mathrm{G}$. Then Lemma 4.5 implies that

$$\mathbf{\Omega}(m) = \bigcap_{\hat{g}\in\mathfrak{H}^{(m)}}\mathbf{\Omega}(m;\hat{g}).$$

Moreover, we may assume that $\mathfrak{H}^{(m)}$ contains $\hat{I}_n$.

$$\mathbf{\Omega}(m;\hat{I}_n) = \left\{\mathbf{u}\in\mathbf{U}^{-} \ :\ \left|\varpi^{M+m}\mathbf{u}_{ij}\right|\leqslant 1\right\}$$

is an admissible open affinoid subset of $\mathbf{U}^{-}$. $\mathbf{\Omega}(m)$ is the intersection of finitely many rational sub-domains of $\mathbf{\Omega}(m;\hat{I}_n)$:

$$\left\{\mathbf{u}\in\mathbf{\Omega}(m;\hat{I}_n) \ :\ \left|\frac{\varpi^{N(M+m)}\mathbf{u}_{ij}^{N}}{f(\hat{g},\mathbf{u})}\right|\leqslant 1, 1\leqslant i\leqslant j\leqslant n\right\},$$

with $\hat{g}$ ranging on $\mathfrak{H}^{(m)}-\{\hat{I}_n\}$. Therefore $\mathbf{\Omega}(m)$ is an affinoid variety.

We conclude that $\{\mathbf{\Omega}(m)\}_{m=0}^{\infty}$ constitutes an admissible affinoid covering of $\mathbf{\Omega}$ so that $\mathbf{\Omega}$ admits a rigid analytic variety structure (see [1] 9.3). According to [1] 9.1.2 Lemma 3 (compare [1] 9.1.4 Proposition 2), the following proposition implies that $\mathbf{\Omega}$ is an admissible open subset of $\mathbf{U}^{-}$.

Proposition 4.6. *Any morphism from an affinoid variety to* $\mathbf{U}^{-}$ *with image in* $\mathbf{\Omega}$ *factors through some* $\mathbf{\Omega}(m)$.

Proof. The argument is similar to the third proof of [10] §1 Proposition 1.

Let $\mathbf{X}$ be an affinoid variety, $\Phi:\mathbf{X}\to\mathbf{U}^{-}$ a morphism from $\mathbf{X}$ to $\mathbf{U}^{-}$ with image in $\mathbf{\Omega}$. For any $\hat{g}\in\mathfrak{H}_{\mathfrak{o}}$,

$$\mathbf{x}\mapsto\frac{\Phi(\mathbf{x})_{ij}^{N}}{f(\hat{g},\Phi(\mathbf{x}))},\quad 1\leqslant i\leqslant j\leqslant n,$$

are F-rigid analytic functions on $\mathbf{X}$. By the maximum modulus principle ([1] §6.2 Proposition 4 (i)), there exists a positive integer $m_{\hat{g}}$ such that

$$\max_{1\leqslant i\leqslant j\leqslant n}\max_{\mathbf{x}\in\mathbf{X}}\left|\frac{\Phi(\mathbf{x})_{ij}^{N}}{f(\hat{g},\Phi(\mathbf{x}))}\right|\leqslant|\varpi|^{-N(M+m_{\hat{g}})}.$$

In other words, $\Phi(\mathbf{X}) \subset \mathbf{\Omega}(m_{\hat{g}}; \hat{g})$. In view of Lemma 4.5, $m_{\hat{g}}$ can be chosen locally constant. Therefore the compactness of $\mathfrak{H}_0$ implies that there exists a positive integer m such that $\Phi(\mathbf{X}) \subset \mathbf{\Omega}(m)$. Q.E.D.

Finally, we prove that the morphisms of g-translations from $\mathbf{\Omega}(m)$ into $\mathbf{\Omega}$ indeed factor through the same $\mathbf{\Omega}(m')$ for all $g \in \mathrm{G}_0$.

Lemma 4.7. *For any nonnegative integer m, there exists a nonnegative integer m′ such that for all $g \in \mathrm{G}_0$,*

$$g * \mathbf{\Omega}(m) \subset \mathbf{\Omega}(m').$$

Proof. Let $\mathbf{u} \in \mathbf{\Omega}(m)$. Then

$$1 \leqslant |\mathbf{u}| \leqslant |\varpi|^{-M-m}, \tag{4.6}$$

and

$$\frac{|\mathbf{u}|^N}{|f(\hat{g}, \mathbf{u})|} \leqslant |\varpi|^{-N(M+m)} \quad \text{for any } g \in \mathrm{G}_0. \tag{4.7}$$

$g * \mathbf{u} = \mathrm{pr}^{\mathbf{C}}_{\mathbf{U}^-}(g \cdot \mathbf{u})$, and since $\mathrm{pr}^{\mathbf{C}}_{\mathbf{U}^-}$ is F-regular on $\mathbf{C}$, Lemma 2.1 (4) implies that there exist positive integers s and t such that all the entries of

$$\det(g \cdot \mathbf{u})^t f(g \cdot \mathbf{u})^s \cdot g * \mathbf{u} = \det(g)^t f(g \cdot \mathbf{u})^s \cdot g * \mathbf{u}$$

are F-polynomials with variables the entries of $g \cdot \mathbf{u}$. Let D be the highest degree and L an integer such that the absolute values of all the coefficients are bounded by $|\varpi|^L$. Since $g \in \mathrm{G}_0$, the entries of $g \cdot \mathbf{u}$ have absolute values $\leqslant |\mathbf{u}|$ and $|\det(g)| = 1$, then

$$|g * \mathbf{u}| \leqslant \frac{|\varpi|^L |\mathbf{u}|^D}{|f(\hat{g}, \mathbf{u})|^s}. \tag{4.8}$$

It follows from (4.4), (4.8), (4.6) and (4.7) that for any $g_1 \in \mathrm{G}_0$,

$$\begin{aligned} \frac{|g * \mathbf{u}|^N}{|f(\widehat{g_1}, g * \mathbf{u})|} &\leqslant \frac{|\varpi|^{NL} |\mathbf{u}|^{ND}}{|f(\widehat{g_1 g}, \mathbf{u})| |f(\hat{g}, \mathbf{u})|^{Ns-1}} \\ &= |\varpi|^{NL} |\mathbf{u}|^{ND - N^2 s} \frac{|\mathbf{u}|^N}{|f(\widehat{g_1 g}, \mathbf{u})|} \frac{|\mathbf{u}|^{N^2 s - N}}{|f(\hat{g}, \mathbf{u})|^{Ns-1}}, \\ &\leqslant |\varpi|^{N(L - \max\{D, Ns\}(M+m))}. \end{aligned}$$

Therefore $g * \mathbf{\Omega}(m) \subset \mathbf{\Omega}(m')$ for any $m' \geqslant -M - L + \max\{D, Ns\}(M+m)$. Q.E.D.

4.4. Rigid analytic functions on $\mathbf{\Omega}$. Let $\mathscr{O}(\mathbf{\Omega}(m))$ denote the space of F-rigid analytic functions on $\mathbf{\Omega}(m)$. Then $\mathscr{O}(\mathbf{\Omega}(m))$ is an F-affinoid algebra with the supremum norm.

Let $\mathscr{O}(\mathbf{\Omega})$ be the F-algebra of F-rigid analytic functions on $\mathbf{\Omega}$, that is, the projective limit of $\mathscr{O}(\mathbf{\Omega}(m))$,

$$\mathscr{O}(\mathbf{\Omega}) := \varprojlim_m \mathscr{O}(\mathbf{\Omega}(m)).$$

$\mathscr{O}(\mathbf{\Omega})$ is endowed with the projective limit topology.

From the construction of $\mathbf{\Omega}(m)$ we see that the F-affinoid algebra $\mathscr{O}(\mathbf{\Omega}(m))$ is equal to

$$F\left\langle \varpi^{M+m}\mathbf{u}_{ij}, \frac{\varpi^{N(M+m)}\mathbf{u}_{ij}^{N}}{f(\hat{g},\mathbf{u})} \;:\; 1 \leqslant i \leqslant j \leqslant n,\ \hat{g} \in \mathfrak{H}^{(m)} - \{\hat{I}_n\} \right\rangle. \tag{4.9}$$

Therefore $\psi \in \mathscr{O}(\mathbf{\Omega}(m))$ has an expansion in the following form that converges with respect to the supremum norm $\| \; \|_{\mathscr{O}(\mathbf{\Omega}(m))}$:

$$\psi(\mathbf{u}) = \sum_{(\ell_{\hat{g}})\in(\mathbb{N}_0)^{\mathfrak{H}^{(m)}}} P_{(\ell_{\hat{g}})}(\mathbf{u}) \prod_{\hat{g}\in\mathfrak{H}^{(m)}} f(\hat{g},\mathbf{u})^{-\ell_{\hat{g}}}, \tag{4.10}$$

where $P_{(\ell_{\hat{g}})}(\mathbf{u})$ are polynomials in the coordinates of $\mathbf{u}$ with coefficients in F.

In view of (4.5), the assumption that $\mathfrak{H}^{(m)}$ is contained in $\mathfrak{H}_0$ is quite artificial, and it is more convenient and natural to choose $\mathfrak{H}^{(m)}$ to be an arbitrary finite subset of $\mathfrak{H}$ whenever we consider the expansion of $\psi \in \mathscr{O}(\mathbf{\Omega}(m))$.

$\psi \in \mathscr{O}(\mathbf{\Omega})$ may be considered as an F^{alg}-valued function on $\mathbf{\Omega}$ such that, restricting on each $\mathbf{\Omega}(m)$, ψ has an expansion (4.10) that converges with respect to $\| \; \|_{\mathscr{O}(\mathbf{\Omega}(m))}$. In particular, $f(\hat{g},\mathbf{u})^{-1} \in \mathscr{O}(\mathbf{\Omega})$ for any $\hat{g} \in \mathfrak{H}$.

Since all the generators of $\mathscr{O}(\mathbf{\Omega}(m))$ in (4.9) are F-rigid analytic functions on $\mathbf{\Omega}(m')$ for any $m' \geqslant m$ and therefore on $\mathbf{\Omega}$, we obtain the following proposition.

Proposition 4.8.
(1) $\mathbf{\Omega}$ *is a Stein space, that is, the image of* $\mathscr{O}(\mathbf{\Omega}(m+1))$ *under the transition homomorphism in* $\mathscr{O}(\mathbf{\Omega}(m))$ *is dense for any nonnegative integer m.*
(2) *The image of* $\mathscr{O}(\mathbf{\Omega})$ *under the transition homomorphism in* $\mathscr{O}(\mathbf{\Omega}(m))$ *is dense.*

Let $\mathscr{O}_K(\mathbf{\Omega}(m))$ and $\mathscr{O}_K(\mathbf{\Omega})$ denote $\mathscr{O}(\mathbf{\Omega}(m))\hat{\otimes}_F K$ and $\mathscr{O}(\mathbf{\Omega})\hat{\otimes}_F K$ respectively. If we let $\mathbf{\Omega}_K(m)$ and $\mathbf{\Omega}_K$ denote the extensions of the ground field K/F of $\mathbf{\Omega}(m)$ and $\mathbf{\Omega}$ respectively (see [1] §9.3.6), then $\mathscr{O}_K(\mathbf{\Omega}(m))$ and $\mathscr{O}_K(\mathbf{\Omega})$ are the spaces of K-rigid analytic functions on $\mathbf{\Omega}_K(m)$ and $\mathbf{\Omega}_K$ respectively.

Proposition 4.9. *Let K be spherically complete.* $\mathscr{O}_K(\mathbf{\Omega})$ *is a nuclear K-Fréchet space.*

Proof. By [9] Proposition 19.9, it suffices to prove that all the $\mathscr{O}_K(\mathbf{\Omega}(m))$ constitute a compact projective system.

Consider the $\mathscr{O}_K(\mathbf{\Omega}(m-1))$-norms of the generators of $\mathscr{O}_K(\mathbf{\Omega}(m))$ (see 4.9), then

$$\sup_{\mathbf{u}\in\mathbf{\Omega}(m-1)} \max_{\hat{g}\in\mathfrak{H}^{(m)}-\{\hat{I}_n\}} \max_{1\leqslant i\leqslant j\leqslant n} \left\{ \left|\varpi^{M+m}\mathbf{u}_{ij}\right|, \left|\frac{\varpi^{N(M+m)}\mathbf{u}_{ij}}{f(\hat{g},\mathbf{u})}\right| \right\} \leqslant |\varpi|.$$

[12] Lemma 1.5 implies that the transition homomorphism from $\mathscr{O}_K(\mathbf{\Omega}(m))$ to $\mathscr{O}_K(\mathbf{\Omega}(m-1))$ is compact. Q.E.D.

[11] Theorem 1.3 and Proposition 1.2 imply the following corollary.

Corollary 4.10. *Suppose K is spherically complete. Let* $\mathscr{N}$ *be a closed subspace of* $\mathscr{O}_K(\mathbf{\Omega})$*, then* $\mathscr{N}$ *and* $\mathscr{O}_K(\mathbf{\Omega})/\mathscr{N}$ *are nuclear Fréchet spaces, and their strong duals* $\mathscr{N}_b^*$ *and* $(\mathscr{O}_K(\mathbf{\Omega})/\mathscr{N})_b^*$ *are of compact type.*

5. Holomorphic discrete series $(\mathscr{O}_\sigma(\Omega), \pi_\sigma)$

Let Ω (resp. $\Omega(m)$) denote $\boldsymbol{\Omega}_K(K)$ (resp. $\boldsymbol{\Omega}_K(m)(K)$). Restricting to Ω (resp. $\Omega(m)$), we view K-rigid analytic functions in $\mathscr{O}_K(\boldsymbol{\Omega})$ (resp. $\mathscr{O}_K(\boldsymbol{\Omega}(m))$) as K-valued functions on Ω (resp. $\Omega(m)$), and abbreviate $\mathscr{O}_K(\boldsymbol{\Omega}(m))$ (resp. $\mathscr{O}_K(\boldsymbol{\Omega})$) to $\mathscr{O}(\Omega(m))$ (resp. $\mathscr{O}(\Omega)$).

Let (V, σ) be a d-dimensional K-rational representation of L. σ extends to a representation of P^+.

Let $\mathscr{O}_\sigma(\Omega) := \mathscr{O}(\Omega) \underset{K}{\otimes} V$ and $\mathscr{O}_\sigma(\Omega(m)) := \mathscr{O}(\Omega(m)) \underset{K}{\otimes} V$.

For any $g \in \mathrm{G}$ and $\psi \in \mathscr{O}_\sigma(\Omega)$, let $\pi_\sigma(g)\psi$ be the V-valued function on Ω as follows

$$(\pi_\sigma(g)\psi)(\mathbf{u}) := \sigma(j(g^{-1}, \mathbf{u}))^{-1} \psi(g^{-1} * \mathbf{u}).$$

Lemma 5.1. $\pi_\sigma(g)\psi \in \mathscr{O}_\sigma(\Omega)$.

Proof. Since σ is K-rational, each coordinate of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ is a product of a K-polynomial in the coordinates of $j(g^{-1}, \mathbf{u})$ and a power of $\det(j(g^{-1}, \mathbf{u}))^{-1} = \det(g)$. Note that $j(g^{-1}, \mathbf{u}) = \mathrm{pr}^{\mathbf{C}}_{\mathbf{P}^+}(g^{-1} \cdot \mathbf{u})$, and since $\mathrm{pr}^{\mathbf{C}}_{\mathbf{P}^+}$ is F-regular on $\mathbf{C}$, each coordinate of $j(g^{-1}, \mathbf{u})$ is a product of an F-polynomial in the coordinates of $g^{-1} \cdot \mathbf{u}$ and powers of $\det(g^{-1} \cdot \mathbf{u})^{-1} = \det(g)$ and $f(g^{-1} \cdot \mathbf{u})^{-1}$. Therefore each coordinate of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ has a finite expansion of the form (4.10), and hence belongs to $\mathscr{O}(\Omega)$.

Similarly, the coordinates of $\psi(g^{-1} * \mathbf{u})$ also have expansions of the form (4.10). By Proposition 4.6, for any m, g^{-1}-translation maps $\boldsymbol{\Omega}(m)$ into some $\boldsymbol{\Omega}(m')$, and hence the norm of each coordinate of $\psi(g^{-1} * \mathbf{u})$ on $\boldsymbol{\Omega}(m)$ is bounded by the norm of the corresponding coordinate of ψ on $\boldsymbol{\Omega}(m')$. Therefore $\psi(g^{-1} * \mathbf{u}) \in \mathscr{O}_\sigma(\Omega)$.

We conclude that $\pi_\sigma(g)\psi \in \mathscr{O}_\sigma(\Omega)$. Q.E.D.

It follows from the automorphy relation (4.1) that π_σ is an action of G on $\mathscr{O}_\sigma(\Omega)$.

Definition 5.2. *We call* $(\mathscr{O}_\sigma(\Omega), \pi_\sigma)$ *the* holomorphic (rigid analytic) discrete series representation *of* G.

Lemma 5.3. *Let m and m' be as in Lemma 4.7. Then there exists a constant c depending on σ and m such that*

$$\|\pi_\sigma(g)\psi\|_{\mathscr{O}_\sigma(\Omega(m))} \leqslant c \|\psi\|_{\mathscr{O}_\sigma(\Omega(m'))},$$

for all $g \in \mathrm{G}_0$.

Proof. The proof is similar to the arguments in Lemma 5.1, but instead of Proposition 4.6 we apply Lemma 4.7.

Using the expressions for the coordinates of $\sigma(j(g^{-1}, \mathbf{u}))^{-1}$ in the first paragraph of the proof of Lemma 5.1, we see that their $\mathscr{O}(\Omega(m))$-norms are uniformly bounded on G_0, so there is a constant $c > 0$ such that

$$\max_{g \in \mathrm{G}_0} \max_{\mathbf{u} \in \boldsymbol{\Omega}(m)} \|\sigma(j(g^{-1}, \mathbf{u}))^{-1}\|_{\mathrm{End}(V)} \leqslant c.$$

Consequently,

$$\begin{aligned}
&\max_{g\in \mathrm{G}_\circ} \|\pi_\sigma(g)\psi\|_{\mathscr{O}_\sigma(\Omega(m))}\\
=&\max_{g\in \mathrm{G}_\circ}\max_{\mathbf{u}\in\mathbf{\Omega}(m)} \|(\pi_\sigma(g)\psi)(\mathbf{u})\|_V\\
\leqslant&\max_{g\in \mathrm{G}_\circ}\max_{\mathbf{u}\in\mathbf{\Omega}(m)} \|\sigma(j(g^{-1},\mathbf{u}))^{-1}\|_{\mathrm{End}(V)}\cdot \max_{g\in \mathrm{G}_\circ}\max_{\mathbf{u}\in\mathbf{\Omega}(m)} \|\psi(g^{-1}*\mathbf{u})\|_V\\
\leqslant& c\max_{\mathbf{u}\in\mathbf{\Omega}(m')}\|\psi(\mathbf{u})\|_V\\
=& c\|\psi\|_{\mathscr{O}_\sigma(\Omega(m'))}.
\end{aligned}$$

Q.E.D.

It follows from Lemma 5.3 that, for each m, the map

$$\begin{array}{rcl}\mathrm{G}_\circ\times\mathscr{O}_\sigma(\Omega) & \to & \mathscr{O}_\sigma(\Omega(m))\\ (g,\psi) & \mapsto & (\pi_\sigma(g)\psi)|_{\Omega(m)}.\end{array}$$

is continuous. Since $\mathscr{O}_\sigma(\Omega)$ is the projective limit of $\mathscr{O}_\sigma(\Omega(m))$, we obtain the following corollary.

Corollary 5.4. *$(\mathscr{O}_\sigma(\Omega),\pi_\sigma)$ is a continuous* G*-representation.*

Moreover, we shall prove that the dual representation of π_σ is locally analytic. For this, we recall that a coordinate chart at 1_{G} is obtained from the decomposition of the Bruhat big cell (see (2.2))

$$\mathrm{U}^-\mathrm{U}_{\mathrm{L}}^-\mathrm{T}\mathrm{U}_{\mathrm{L}}^+\mathrm{U}^+\simeq \mathbb{A}_F^{|R|}\times\mathbb{G}_m(F)^{\dim\mathfrak{g}_0}. \tag{5.1}$$

Lemma 5.5. *Let m and m' be as in Lemma 4.7. Let* B *be any parameterized (as in (5.1)) open neighborhood of 1_{G} contained in $\mathrm{G}_\circ$. For any $\psi\in\mathscr{O}_\sigma(\Omega(m'))$, the orbit map*

$$\begin{array}{rcl}\mathrm{B} & \to & \mathscr{O}_\sigma(\Omega(m))\\ g & \mapsto & (\pi_\sigma(g)\psi)|_{\Omega(m)}\end{array}$$

is an $\mathscr{O}_\sigma(\Omega(m))$-valued analytic function, namely, it can be expanded as a convergent power series with variables the coordinate parameters of B *and coefficients in the Banach space $\mathscr{O}_\sigma(\Omega(m))$.*

Proof. Once we have obtained a formal expansion of $\pi_\sigma(g)\psi$ into a power series with variables the coordinate parameters of B and coefficients in $\mathscr{O}_\sigma(\Omega(m))$, Lemma 5.3 would imply that the expansion is indeed convergent. In view of (5.1), it suffices to consider $\pi_\sigma(g)\psi(\mathbf{u})$ for g in $\mathrm{U}^-,\mathrm{U}_{\mathrm{L}}^-$ and T (note that U^+ and $\mathrm{U}_{\mathrm{L}}^+$ are the conjugations of U^- and $\mathrm{U}_{\mathrm{L}}^-$ by the long Weyl element).

Let $u\in\mathrm{U}^-$, then

$$\pi_\sigma(u)\psi(\mathbf{u})=\psi(u^{-1}\cdot\mathbf{u}).$$

Let $\mathscr{O}(\Omega(m))[[u]]$ denote the ring of formal power series $\varphi(u)$ in the coordinates u_α ($\alpha \in R_I^-$) with coefficients in $\mathscr{O}(\Omega(m))$, where $\varphi(u)$ is expressed as

$$\varphi(u) = \sum_{\underline{r} \in \mathbb{N}_0^{R_I^-}} a_{\underline{r}} \cdot \underline{u}^{\underline{r}}, \quad a_{\underline{r}} \in \mathscr{O}(\Omega(m)),\ \underline{u}^{\underline{r}} := \prod_{\alpha \in R_I^-} u_\alpha^{r_\alpha}.$$

If the constant term $a_{\underline{0}}$ is a unit in $\mathscr{O}(\Omega(m))$, then $\varphi(u)$ is invertible in $\mathscr{O}(\Omega(m))[[u]]$. Note that, for $\hat{g} \in \mathfrak{H}$, the constant term in the expansion of $f(\hat{g}, u^{-1} \cdot \mathbf{u})$ is $f(\hat{g}, \mathbf{u})$, and it is invertible in $\mathscr{O}(\Omega(m))$, so $f(\hat{g}, u^{-1} \cdot \mathbf{u})^{-1}$ belongs to $\mathscr{O}(\Omega(m))[[u]]$. Therefore, in view of the expansion form (4.10), the coordinates of $\psi(u^{-1} \cdot \mathbf{u})$ expand into a formal power series in u_α whose coefficients are series in $\mathscr{O}(\Omega(m))$, but it follows from Lemma 5.3 that the coefficients are indeed convergent series in $\mathscr{O}(\Omega(m))$ for $u \in \mathrm{B}$. So each coordinate of $\psi(u^{-1} \cdot \mathbf{u})$ belongs to $\mathscr{O}(\Omega(m))[[u]]$.

For $l \in \mathrm{U}_\mathrm{L}^-$ or T,

$$\pi_\sigma(l)\psi(\mathbf{u}) = \sigma(l)^{-1}\psi(l^{-1} \cdot \mathbf{u} \cdot l).$$

The arguments are similar. Q.E.D.

Corollary 5.6. *Let* $\mathrm{U}_\mathfrak{o}^+ = \mathrm{U}^+ \cap \mathrm{G}_\mathfrak{o}$, *then the power series expansion of*

$$f(j(u^+, \mathbf{u}))^{-1} = f(u^+ \cdot \mathbf{u})^{-1}, \quad u^+ \in \mathrm{U}_\mathfrak{o}^+,$$

on $\mathrm{U}_\mathfrak{o}^+$ *converges in* $\mathscr{O}(\Omega(m))$.

Proof. Since f is an F-rational character on P^+ (Lemma 2.1 (1)), if we put $\sigma = f$ and $\psi \equiv 1$, then $(\pi_f(g^{-1})1)(\mathbf{u}) = f(j(g, \mathbf{u}))^{-1}$. Therefore our assertion follows from Lemma 5.5. Q.E.D.

Now consider the dual representation π_σ^* of G on $\mathscr{O}_\sigma(\Omega)_b^* \cong \varinjlim_m \mathscr{O}(\Omega(m))_b^*$. The transition homomorphisms $\mathscr{O}_\sigma(\Omega(m))_b^* \to \mathscr{O}_\sigma(\Omega)_b^*$ are injective (see Proposition 4.8 (2)). Lemma 4.7 implies that, for any $g \in \mathrm{G}_\mathfrak{o}$, $\pi_\sigma^*(g)$ maps $\mathscr{O}(\Omega(m))_b^*$ into $\mathscr{O}(\Omega(m'))_b^*$ via

$$\langle \psi,\ \pi_\sigma^*(g)\mu \rangle = \left\langle (\pi_\sigma(g^{-1})\psi)|_{\Omega(m)},\ \mu \right\rangle, \quad \mu \in \mathscr{O}_\sigma(\Omega(m))^*, \psi \in \mathscr{O}_\sigma(\Omega(m')).$$

We deduce from Lemma 5.5 that, for any $\mu \in \mathscr{O}_\sigma(\Omega(m))^*$, the orbit map

$$\begin{array}{ccc} \mathrm{B}^{-1} & \to & \mathscr{O}_\sigma(\Omega(m'))_b^* \\ g & \mapsto & \pi_\sigma^*(g)\mu \end{array}$$

is an $\mathscr{O}_\sigma(\Omega(m'))_b^*$-valued analytic function. Therefore we obtain the following corollary.

Corollary 5.7. $(\mathscr{O}_\sigma(\Omega)_b^*, \pi_\sigma^*)$ *is locally analytic.*

6. Duality

In the following, we assume that K is spherically complete. Let (V, σ) be a d-dimensional K-rational representation of L. We choose a basis $v_1, \cdots, v_d$ of V and denote by $v_1^*, \cdots, v_d^*$ the corresponding dual basis of the dual space V^*. (V^*, σ^*) denotes the dual representation of (V, σ).

6.1. The duality operator I_σ. For $\mathbf{u} \in \Omega$ and $v^* \in V^*$, let $\varphi_{\mathbf{u},v^*}$ be the V^*-valued locally analytic function on $\mathfrak{H}$:

$$\varphi_{\mathbf{u},v^*}(\hat{g}) := \sigma^*(j(\hat{g}, \mathbf{u}))v^*.$$

In view of (4.2), $\varphi_{\mathbf{u},v^*}$ belongs to $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)$. Let $B^0_{\sigma^*}(\mathfrak{H}, V^*)$ be the subspace of $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)$ spanned by $\varphi_{\mathbf{u},v^*}$, $B_{\sigma^*}(\mathfrak{H}, V^*)$ the closure of $B^0_{\sigma^*}(\mathfrak{H}, V^*)$. From (4.1), we see that $B^0_{\sigma^*}(\mathfrak{H}, V^*)$ and therefore $B_{\sigma^*}(\mathfrak{H}, V^*)$ are G-invariant.

For any continuous linear functional $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, we define a V-valued function on Ω:

$$I_\sigma(\xi)(\mathbf{u}) := \sum_{k=1}^{d} \langle \varphi_{\mathbf{u},v_k^*}, \xi \rangle v_k, \quad \mathbf{u} \in \Omega.$$

$I_\sigma(\xi)$ is independent of the choice of the basis $\{v_k\}_{k=1}^d$. Evidently, I_σ is injective.

Lemma 6.1. *I_σ is* G*-equivariant, that is,*

$$I_\sigma(T^*_{\sigma^*}(g)\xi) = \pi_\sigma(g) I_\sigma(\xi),$$

for any $g \in$ G.

Proof.

$$\begin{aligned}
I_\sigma(T^*_{\sigma^*}(g)\xi)(\mathbf{u}) &= \sum_{k=1}^{d} \langle \varphi_{\mathbf{u},v_k^*}, T^*_{\sigma^*}(g)\xi \rangle v_k = \sum_{k=1}^{d} \langle T_{\sigma^*}(g^{-1})\varphi_{\mathbf{u},v_k^*}, \xi \rangle v_k \\
&= \sum_{k=1}^{d} \langle \sigma^*(j(? \cdot g^{-1}, \mathbf{u}))v_k^*, \xi \rangle v_k \\
&= \sigma(j(g^{-1}, \mathbf{u}))^{-1} \left(\sum_{k=1}^{d} \langle \sigma^*(j(?, g^{-1} * \mathbf{u}))v_{k;g}^*, \xi \rangle v_{k;g} \right) \quad \text{(see (4.1))} \\
&= (\pi_\sigma(g) I_\sigma(\xi))(\mathbf{u}),
\end{aligned}$$

where $v_{k;g} = \sigma(j(g^{-1}, \mathbf{u}))v_k$, and similarly $v_{k;g}^* = \sigma^*(j(g^{-1}, \mathbf{u}))v_k^*$. $\{v_{k;g}\}_{k=1}^d$ and $\{v_{k;g}^*\}_{k=1}^d$ are dual to each other. Q.E.D.

Proposition 6.2.

(1) *For any continuous linear functional $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, $I_\sigma(\xi)$ is a V-valued rigid analytic function on Ω.*

(2) *I_σ is a continuous homomorphism of* G*-representations from $(B_{\sigma^*}(\mathfrak{H}, V^*)^*_b, T^*_{\sigma^*})$ to $(\mathscr{O}_\sigma(\Omega), \pi_\sigma)$.*

Proof. *Step 1.* We denote by i the inclusion: $B_{\sigma^*}(\mathfrak{H}, V^*) \hookrightarrow C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)$, i^* its adjoint operator. Because of our assumption that K is spherically complete, the Hahn-Banach Theorem ([9] Corollary 9.4) implies that i^* is surjective. Since $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)^*_b$ and $B_{\sigma^*}(\mathfrak{H}, V^*)^*_b$ are both Fréchet spaces (Corollary 3.6), the open mapping theorem ([9] Proposition 8.6) implies that i^* is open. Therefore the continuity of $I_\sigma \circ i^*$ implies that of I_σ. Consequently, (1) and (2) are equivalent to:

(1′) $I_\sigma \circ i^*(\xi) \in \mathscr{O}_\sigma(\Omega)$ for any $\xi \in C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)^*$;
(2′) $I_\sigma \circ i^* : (C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)^*_b, T^*_{\sigma^*}) \to (\mathscr{O}_\sigma(\Omega), \pi_\sigma)$ is a continuous homomorphism of G-representations.

Since G-equivariance is proved in Lemma 6.1, for (2′) it remains to show the continuity of $I_\sigma \circ i^*$.

For convenience, we still denote $I_\sigma \circ i^*$ by I_σ.

Step 2. Let $\{\overline{\mathfrak{U}}_\kappa\}_\kappa$ be a finite disjoint open covering of $\overline{\mathfrak{H}}$ satisfying:
1. $\mathrm{U}^+_{\mathfrak{o}} \in \{\overline{\mathfrak{U}}_\kappa\}_\kappa$ (note that the open subscheme $\mathrm{P}^-\backslash \mathrm{C}$ of $\overline{\mathfrak{H}}$ is identified with U^+);
2. each $\overline{\mathfrak{U}}_\kappa$ is (right) translated into $\mathrm{U}^+_{\mathfrak{o}}$ by some $g_\kappa \in \mathrm{G}$.
Let $\mathfrak{U}_\kappa$ be the preimage of $\overline{\mathfrak{U}}_\kappa$ under $\mathrm{pr}^{\mathfrak{H}}_{\overline{\mathfrak{H}}}$.

For $\xi \in C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)^*$, we write $I_\sigma(\xi)$ in integral:

$$I_\sigma(\xi)(\mathbf{u}) = \sum_{k=1}^{d} \int_{\mathfrak{H}} \varphi_{\mathbf{u}; v_k^*} d\xi \cdot v_k = \sum_{k=1}^{d} \sum_{\kappa} \int_{\mathfrak{U}_\kappa} \varphi_{\mathbf{u}; v_k^*} d\xi \cdot v_k$$
$$= \sum_{\kappa} \pi_\sigma(g_\kappa) \Big(\sum_{k=1}^{d} \int_{\mathfrak{U}_\kappa \cdot g_\kappa} \varphi_{\mathbf{u}; v^*_{k; g_\kappa}} d(T^*_{\sigma^*}(g_\kappa^{-1})\xi) \cdot v_{k; g_\kappa} \Big),$$

where $v_{k;g_\kappa} = \sigma(j(g_\kappa^{-1}, \mathbf{u}))v_k$ is defined in the proof of Lemma 6.1. Therefore it suffices to consider

$$\sum_{k=1}^{d} \int_{\mathfrak{U}} \varphi_{\mathbf{u}; v_k^*} d\xi' \cdot v_k, \tag{6.1}$$

where $\mathfrak{U}$ ranges on $\{\mathfrak{U}_\kappa \cdot g_\kappa\}_\kappa$ and ξ' is the image of ξ under $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)^*_b \to C^{\mathrm{an}}_{\sigma^*}(\mathfrak{U}, V^*)^*_b$.

For the open subset $\overline{\mathfrak{U}} = \mathrm{pr}^{\mathfrak{H}}_{\overline{\mathfrak{H}}}(\mathfrak{U})$ of $\mathrm{U}^+_{\mathfrak{o}}$, we have the isomorphism induced from a locally analytic section ι of $\mathrm{pr}^{\mathfrak{U}}_{\overline{\mathfrak{U}}}$ (see Lemma 3.5 (3)):

$$C^{\mathrm{an}}_{\sigma^*}(\mathfrak{U}, V^*)^*_b \simeq C^{\mathrm{an}}(\overline{\mathfrak{U}}, V^*)^*_b. \tag{6.2}$$

Then (6.1) is equal to

$$\bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})(\mathbf{u}) := \sum_{k=1}^{d} \int_{\overline{\mathfrak{U}}} (\sigma^*(j(u^+, \mathbf{u}))v_k^*) \, d\bar{\xi}(u^+) \cdot v_k,$$

where $\bar{\xi}$ is the image of ξ' in $C^{\mathrm{an}}(\overline{\mathfrak{U}}, V^*)^*_b$ via the isomorphism (6.2).

Therefore it suffices to prove that $\bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})$ is rigid analytic on $\Omega(m)$, and that the map

$$\begin{array}{ccc} C^{\mathrm{an}}(\overline{\mathfrak{U}}, V^*)^*_b & \to & \mathscr{O}_\sigma(\Omega(m)) \\ \bar{\xi} & \mapsto & \bar{I}_{\sigma, \overline{\mathfrak{U}}}(\bar{\xi})|_{\Omega(m)} \end{array}$$

is continuous for all m.

Step 3. Since σ^* is K-rational, using the same arguments in the proof of Lemma 5.1 and applying Corollary 5.6, we obtain an expansion

$$\sigma^*(j(u^+, \mathbf{u}))v_k^* = \sum_{\ell=1}^{d} \Big(\sum_{\underline{r} \in \mathbb{N}_0^{R^+_I}} a_{\underline{r}, k\ell}(\mathbf{u}) \cdot (\underline{u}^+)^{\underline{r}} \Big) v_\ell^*,$$

with $a_{\underline{r},k\ell} \in \mathscr{O}(\Omega(m))$ such that

$$\lim_{|\underline{r}|\to\infty} \|a_{\underline{r},k\ell}\|_{\mathscr{O}(\Omega(m))} \cdot \|(\underline{u}^+)^{\underline{r}}\|_{C^{\mathrm{an}}(\mathrm{U}_0^+)} = 0, \tag{6.3}$$

and moreover, there is a constant $c' > 0$, depending only on m, σ and $\{v_k\}_{k=1}^d$, such that

$$\|a_{\underline{r},k\ell}\|_{\mathscr{O}(\Omega(m))} \cdot \|(\underline{u}^+)^{\underline{r}}\|_{C^{\mathrm{an}}(\mathrm{U}_0^+)} \leqslant c'. \tag{6.4}$$

Then

$$\bar{I}_{\sigma,\overline{\mathfrak{U}}}(\bar{\xi})(\mathbf{u}) = \sum_{k=1}^d \Big(\sum_{\ell=1}^d \sum_{\underline{r}} \int_{\overline{\mathfrak{U}}} (\underline{u}^+)^{\underline{r}} \cdot v_\ell^* d\bar{\xi}(u^+) \cdot a_{\underline{r},k\ell}(\mathbf{u}) \Big) v_k. \tag{6.5}$$

We have

$$\left| \int_{\overline{\mathfrak{U}}} (\underline{u}^+)^{\underline{r}} \cdot v_\ell^* d\bar{\xi}(u^+) \right| \leqslant \|(\underline{u}^+)^{\underline{r}}\|_{C^{\mathrm{an}}(\mathrm{U}_0^+)} \cdot \|v_\ell^*\|_{V^*} \cdot \|\bar{\xi}\|_{C^{\mathrm{an}}(\overline{\mathfrak{U}},V^*)_b^*}. \tag{6.6}$$

(6.3) and (6.6) imply that the expansion (6.5) of $\bar{I}_{\sigma,\overline{\mathfrak{U}}}(\bar{\xi})$ converges in $\mathscr{O}_\sigma(\Omega(m))$.

(6.4) and (6.6) imply

$$\left\|\bar{I}_{\sigma,\overline{\mathfrak{U}}}(\bar{\xi})\right\|_{\mathscr{O}_\sigma(\Omega(m))} \leqslant \max_{1\leqslant k,\ell\leqslant d} c' \|v_\ell^*\|_{V^*} \|v_k\|_V \cdot \|\bar{\xi}\|_{C^{\mathrm{an}}(\overline{\mathfrak{U}},V^*)_b^*},$$

and therefore the continuity follows. Q.E.D.

6.2. The duality operator J_σ. Let $\mathscr{N}_\sigma(\Omega)$ denote the image of I_σ.

We consider J_σ, the adjoint operator of I_σ, which is an injective continuous linear operator from $\mathscr{N}_\sigma(\Omega)_b^*$ to $(B_{\sigma^*}(\mathfrak{H}, V^*)_b^*)_b^* \cong B_{\sigma^*}(\mathfrak{H}, V^*)$ ($B_{\sigma^*}(\mathfrak{H}, V^*)$ is reflexive according to Corollary 3.6).

For any $\mu \in \mathscr{N}_\sigma(\Omega)^*$ and $\xi \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$, we have

$$\langle J_\sigma(\mu), \xi\rangle = \langle I_\sigma(\xi), \mu\rangle. \tag{6.7}$$

For $\hat{g} \in \mathfrak{H}$ and $v \in V$, we define the Dirac distribution $\xi_{\hat{g},v} \in B_{\sigma^*}(\mathfrak{H}, V^*)^*$ as follows:

$$\langle \varphi, \xi_{\hat{g},v}\rangle = \langle v, \varphi(\hat{g})\rangle_V, \quad \varphi \in B_{\sigma^*}(\mathfrak{H}, V^*),$$

and a V-valued rigid analytic function $\psi_{\hat{g},v}$ on Ω:

$$\psi_{\hat{g},v}(\mathbf{u}) := \sigma(j(\hat{g}, \mathbf{u}))^{-1} v.$$

Lemma 6.3.

$$I_\sigma(\xi_{\hat{g},v}) = \psi_{\hat{g},v}.$$

Proof. This is straightforward from definitions.

$$\begin{aligned} I_\sigma(\xi_{\hat{g},v})(\mathbf{u}) &= \sum_{k=1}^d \langle \varphi_{\mathbf{u},v_k^*}, \xi_{\hat{g},v}\rangle v_k = \sum_{k=1}^d \langle v, \varphi_{\mathbf{u},v_k^*}(\hat{g})\rangle_V v_k = \sum_{k=1}^d \langle v, \sigma^*(j(\hat{g},\mathbf{u}))v_k^*\rangle_V v_k \\ &= \sum_{k=1}^d \langle \sigma(j(\hat{g},\mathbf{u}))v, v_k^*\rangle_V v_k = \sigma(j(\hat{g},\mathbf{u}))v = \psi_{\hat{g},v}(\mathbf{u}) \end{aligned}$$

Q.E.D.

Then we obtain a formula for J_σ.

Proposition 6.4. *For any continuous linear functional $\mu \in \mathscr{N}_\sigma(\Omega)^*$, we have*

$$J_\sigma(\mu)(\hat{g}) = \sum_{k=1}^{d} \langle \psi_{\hat{g}, v_k}, \mu \rangle v_k^*. \tag{6.8}$$

Proof. This is straightforward from Lemma 6.3 and (6.7). Indeed,

$$\begin{aligned}\sum_{k=1}^{r} \langle \psi_{\hat{g}, v_k}, \mu \rangle v_k^* = \sum_{k=1}^{r} \langle I_\sigma(\xi_{\hat{g}, v_k}), \mu \rangle v_k^* &= \sum_{k=1}^{r} \langle J_\sigma(\mu), \xi_{\hat{g}, v_k} \rangle v_k^* \\ &= \sum_{k=1}^{r} \langle v_k, J_\sigma(\mu)(\hat{g}) \rangle_V \cdot v_k^* = J_\sigma(\mu)(\hat{g}).\end{aligned}$$

Q.E.D.

6.3. The image of I_σ. Let $\mathscr{N}_\sigma^0(\Omega)$ denote the subspace of $\mathscr{O}_\sigma(\Omega)$ spanned by $\psi_{\hat{g},v}$ for all $\hat{g} \in \mathfrak{H}$ and $v \in V$. Then it follows from (4.1) that $\mathscr{N}_\sigma^0(\Omega)$ is G-invariant, and Lemma 6.3 implies $\mathscr{N}_\sigma^0(\Omega) \subset \mathscr{N}_\sigma(\Omega)$.

From (6.8), we see that J_σ factors through $\mathscr{N}_\sigma^0(\Omega)^*$, and (6.8) defines an injective map from $\mathscr{N}_\sigma^0(\Omega)_b^*$ into $B_{\sigma^*}(\mathfrak{H}, V^*)$. Since J_σ is injective and the natural map $\mathscr{N}_\sigma(\Omega)_b^* \rightarrow \mathscr{N}_\sigma^0(\Omega)_b^*$ is surjective (the Hahn-Banach Theorem), $\mathscr{N}_\sigma^0(\Omega)_b^* = \mathscr{N}_\sigma(\Omega)_b^*$. Therefore the Hahn-Banach Theorem implies the following lemma.

Lemma 6.5. *$\mathscr{N}_\sigma^0(\Omega)$ is dense in $\mathscr{N}_\sigma(\Omega)$.*

Theorem 6.6.
(1) *I_σ is an isomorphism from $B_{\sigma^*}(\mathfrak{H}, V^*)_b^*$ to $\mathscr{N}_\sigma(\Omega)$.*
(2) *$\mathscr{N}_\sigma(\Omega)$ is the closure of $\mathscr{N}_\sigma^0(\Omega)$ in $\mathscr{O}_\sigma(\Omega)$.*

Proof. Let ι be a locally analytic section of $\mathrm{pr}^{\mathfrak{H}}_{\overline{\mathfrak{H}}}$, and denote $\mathfrak{K} = \iota(\overline{\mathfrak{H}})$.

1. Let $\mathscr{N}_\sigma^0(\Omega(m))$ be the image of $\mathscr{N}_\sigma^0(\Omega)$ in $\mathscr{O}_\sigma(\Omega(m))$.

Since $\psi_{\hat{g}, v_k} = \pi_\sigma(g^{-1})v_k$, we see that the map

$$\begin{aligned} \mathfrak{H} &\rightarrow \mathscr{O}_\sigma(\Omega(m)) \\ \hat{g} &\mapsto \psi_{\hat{g}, v_k}, \end{aligned}$$

is locally analytic (Lemma 5.5). Since $\mathfrak{K}$ is compact, $\rho_m = \min_{1 \leqslant k \leqslant d} \min_{\hat{g} \in \mathfrak{K}} \|\psi_{\hat{g}, v_k}\|_{\mathscr{O}_\sigma(\Omega(m))}$ is positive. Let $\mathscr{L}$ denote the lattice $\sum_{k=1}^{d} \sum_{\hat{g} \in \mathfrak{K}} \mathfrak{o}_K \cdot \psi_{\hat{g}, v_k}$ in $\mathscr{N}_\sigma^0(\Omega)$. Then, for each m, the image of $\mathscr{L}$ in $\mathscr{N}_\sigma^0(\Omega(m))$ contains the ball of radius ρ_m centered at zero, and therefore the interior of $\mathscr{L}$ is a nontrivial open lattice.

2. According to Lemma 3.5, ι induces an isomorphism ι° between $C^{\mathrm{an}}_{\sigma^*}(\mathfrak{H}, V^*)$ and $C^{\mathrm{an}}(\overline{\mathfrak{H}}, V^*)$, and hence an isomorphism between $B_{\sigma^*}(\mathfrak{H}, V^*)$ and its image, denoted by $B(\overline{\mathfrak{H}}, V^*)$.

Let $\mathcal{I}$ be any (finite) disjoint open chart covering $\{\overline{\mathfrak{U}}_\kappa\}_\kappa$ of $\overline{\mathfrak{H}}$. We recall that $C^{\mathrm{an}}(\overline{\mathfrak{H}}, V^*)$ is defined to be the inductive limit, indexed with all the $\mathcal{I}$, of the K-Banach algebras $E_{\mathcal{I}}(\overline{\mathfrak{H}}, V^*) = \prod_\kappa O(\overline{\mathfrak{U}}_\kappa, V^*)$, where $O(\overline{\mathfrak{U}}_\kappa, V^*)$ denotes the space of K-analytic functions on $\overline{\mathfrak{U}}_\kappa$ (cf. [2] 2.1.10 and [11] §2). The inductive limit structure is naturally induced onto $B(\overline{\mathfrak{H}}, V^*)$, say $B(\overline{\mathfrak{H}}, V^*) = \varinjlim_{\mathcal{I}} E_{\mathcal{I}}(\overline{\mathfrak{H}}, V^*)$. Moreover, the strong dual space $B(\overline{\mathfrak{H}}, V^*)^*_b$ is the projective limit of $E_{\mathcal{I}}(\overline{\mathfrak{H}}, V^*)^*_b$.

3. Consider

$$\begin{aligned} (\iota^{\circ -1})^* \circ I_\sigma^{-1}|_{\mathscr{N}_\sigma^0(\Omega)} : \mathscr{N}_\sigma^0(\Omega) &\to B(\overline{\mathfrak{H}}, V^*)^*_b \\ \psi_{\hat{g},v} &\mapsto (\iota^{\circ -1})^*(\xi_{\hat{g},v}). \end{aligned}$$

Let $\hat{g} \in \mathfrak{R}$.

$$\begin{aligned} \left\|(\iota^{\circ -1})^*(\xi_{\hat{g},v})\right\|_{E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*)^*_b} &= \max_{\overline{\varphi} \in E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*)} \frac{\langle \overline{\varphi},\ (\iota^{\circ -1})^*(\xi_{\hat{g},v})\rangle}{\|\overline{\varphi}\|_{E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*)}} \\ &= \max_{\varphi \in\ \iota^{\circ -1}(E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*))} \frac{\langle \varphi,\ \xi_{\hat{g},v}\rangle}{\|\iota^\circ(\varphi)\|_{E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*)}} \\ &= \max_{\varphi \in\ \iota^{\circ -1}(E_{\mathcal{I}}(\overline{\mathfrak{H}},V^*))} \frac{\langle v, \varphi(\hat{g})\rangle_V}{\max\limits_{\hat{g}' \in \mathfrak{R}} \|\varphi(\hat{g}')\|_{V^*}} \\ &\leqslant \|v\|_V. \end{aligned}$$

Therefore the image of $\mathscr{L}$ under $(\iota^{\circ -1})^* \circ I_\sigma^{-1}|_{\mathscr{N}_\sigma^0(\Omega)}$ in $B(\overline{\mathfrak{H}}, V^*)^*_b$ is bounded, since its image in $E_{\mathcal{I}}(\overline{\mathfrak{H}}, V^*)^*_b$ are all norm-bounded by $\max\limits_{1\leqslant k\leqslant d} \|v_k\|_V$. Because $\mathscr{N}_\sigma^0(\Omega)$ is metrizable, it is bornological ([9] Proposition 6.14), and therefore $I_\sigma^{-1}|_{\mathscr{N}_\sigma^0(\Omega)}$ is continuous ([9] Proposition 6.13). Therefore I_σ induces an isomorphism between $I_\sigma^{-1}(\mathscr{N}_\sigma^0(\Omega))$ and $\mathscr{N}_\sigma^0(\Omega)$, and consequently I_σ induces an isomorphism between their completions, which, in view of Lemma 6.5, must be $B_{\sigma^*}(\mathfrak{H}, V^*)^*_b$ and $\mathscr{N}_\sigma(\Omega)$ respectively. Q.E.D.

Corollary 6.7. *J_σ is an isomorphism of* G*-representations from* $(\mathscr{N}_\sigma(\Omega)^*_b, \pi_\sigma^*)$ *to* $(B_{\sigma^*}(\mathfrak{H}, V^*), T_{\sigma^*})$.

7. Concluding remarks

In [7] §3 we briefly reviewed Morita's theory of $\mathrm{SL}(2, F)$ and discussed the relation between I_σ and Morita's duality and Casselman's operator for

$$\sigma_s \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = z^s$$

with s a positive integer (for s non-positive, I_σ is an isomorphism between two $(-s+1)$-dimensional G-representations, which is of less interest).

To illustrate this connection, we consider the special case $s = 2$. $\mathscr{O}_{\sigma_2}(\Omega)$ is canonically isomorphic to the space $\Omega^1(\Omega)$ of holomorphic 1-forms on the upper half plane $\Omega \cong K - F = \mathbb{P}^1(K) - \mathbb{P}^1(F)$ via $\psi(\mathbf{u}) \mapsto \psi(\mathbf{u})d\mathbf{u}$. It may be shown that $\mathscr{N}_{\sigma_2}(\Omega)$ corresponds to the subspace of $\Omega^1(\Omega)$ with zero residue at each point of $\mathbb{P}^1(F)$ (see [4] and [7]). On the other hand, $C^{\mathrm{an}}_{\sigma_0}(\mathfrak{H}) \cong C^{\mathrm{an}}(\overline{\mathfrak{H}})$ with $\overline{\mathfrak{H}} \cong \mathbb{P}^1(F)$, and we denote $D_0 = C^{\mathrm{an}}(\mathbb{P}^1(F))$. D_0 has two closed G-invariant subspaces, the spaces P_0 and P_0^{loc} consisting of constants and locally constant functions on $\mathbb{P}^1(F)$ respectively. The classical Morita's duality is established via residues. More precisely, for each $\psi \in \mathscr{O}_{\sigma_2}(\Omega)$, define a linear functional $M_2(\psi)$ of D_0 by

$$\langle \varphi, M_2(\psi) \rangle = \text{ the sum of residues of the 1-form } \varphi(u)\psi(u)\, du \text{ on } \mathbb{P}^1(F).$$

Morita's duality M_2 induces G-isomorphisms $\mathscr{O}_{\sigma_2}(\Omega) \cong (D_0/P_0)^*_b$ and $\mathscr{N}_{\sigma_2}(\Omega) \cong (D_0/P_0^{\mathrm{loc}})^*_b$. Moreover, Casselman's intertwining operator

$$S_0 : \varphi \mapsto d\varphi$$

induces a G-isomorphism between D_0/P_0^{loc} and the space D_{-2} of locally analytic 1-forms on $\mathbb{P}^1(F)$, a space that is isomorphic to $C^{\mathrm{an}}_{\sigma_{-2}}(\mathfrak{H})$ (see [5] and [7]). The connection between our duality operator I_{σ_2} and Morita's duality M_2 was found in [7] Theorem 3.6 as the following commutative diagram

$$\begin{tikzcd}
\mathscr{N}_{\sigma_2}(\Omega) \arrow[rr, "M_2"] & & (D_0/P_0^{\mathrm{loc}})^*_b \\
B_{\sigma_{-2}}(\mathfrak{H})^*_b \arrow[u, "I_{\sigma_2}"] & = C^{\mathrm{an}}_{\sigma_{-2}}(\mathfrak{H})^*_b \cong & (D_{-2})^*_b \arrow[u, "S_0^*"']
\end{tikzcd}$$

A generalization of Morita's duality seems quite hard in view of its analytic construction via residues. The first step towards this would be finding other closed subrepresentations of $(C^{\mathrm{an}}_\sigma(\mathfrak{H}, V), T_\sigma)$ and $(\mathscr{O}_\sigma(\Omega), \pi_\sigma)$. This work is done completely for $\mathrm{SL}(2, F)$ in Morita and Murase's [4], [5] and [6], where the complete classifications of the sub-quotient spaces of holomorphic discrete series and the principal series are conjectured and claimed (Morita attempted to prove this, but his proof contained a serious gap).

A further question is on the irreducibility. For this, we conjecture that $(\mathscr{N}_\sigma(\Omega), \pi_\sigma)$ and $(B_{\sigma^*}(\mathfrak{H}, V^*), T_{\sigma^*})$ are topologically irreducible G-representations if σ is irreducible. For $\mathrm{SL}(2, F)$, this conjecture was claimed in [6] Theorem 1 (i) and a proof for $F = \mathbb{Q}_p$ was given by Schneider and Teitelbaum in [11].

References

[1] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean analysis: a systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften, 261, Springer-Verlag, Berlin-New York, 1984.

[2] C. T. Féaux de Lacroix, *Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper*, Thesis, Köln 1997, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23 (1999) 1-111 .

[3] J. C. Jantzen, *Representations of Algebraic Groups, 2nd Edition*, Mathematical Surveys and Monographs, 107, American Mathematical Society, 2003.

[4] Y. Morita, A. Murase, *Analytic representations of* SL_2 *over a* $\mathfrak{p}$*-adic number field*, J. Fac. Sci. Univ. Tokyo, Sect. IA, 28 (1982) 891-905.

[5] Y. Morita, *Analytic representations of* SL_2 *over a* $\mathfrak{p}$*-adic number field, II*, *Automorphic forms of several variables : Taniguchi Symposium, Katata, 1983*, Prog. Math., 46, 282-297, Birkhaüser, Boston, 1984.

[6] Y. Morita, *Analytic representations of* SL_2 *over a* $\mathfrak{p}$*-adic number field, III*, Automorphic Forms and Number Theory, Adv. Studies Pure Math., 7, 185-222, Tokyo: Kinokuniya, 1985.

[7] Z. Qi, C. Yang, *Morita's Theory for the Symplectic Groups*, Int. J. Number Theory, 7 (2011) 2115-2137.

[8] P. Schneider, *The cohomology of local systems on p-adically uniformized varieties*, Math. Ann. 293 (1992) 623-650.

[9] P. Schneider, *Nonarchimedean Functional Analysis*, Springer-Verlag, Berlin-New York, 2001.

[10] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Invent. Math. 105, no. 1 (1991) 47-122 .

[11] P. Schneider, J. Teitelbaum, *Locally analytic distributions and p-adic representation theory, with application to* GL_2, J. Amer. Math. Soc., 15 (2002) 443-468.

[12] P. Schneider, J. Teitelbaum, *p-adic boundary values*, Asterérisque, 278 (2002) 51-125.

[13] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, 1967.

[14] M. van der Put, H. Voskuil, *Symmetric spaces associated to split algebraic groups over a local field*, J. reine angew. Math., 433 (1992) 69-100.