A bijective proof of $f_{n+4} + f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} + 3$

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Abstract

In *Proofs that Really Count*, Benjamin and Quinn mentioned that there was no known bijective proof for the identity $f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$ for $n \geq 0$, where f_k is the k-th Fibonacci number. In this paper, we interpret f_k as the cardinality of the set F_k consisting of all ordered lists of 1's and 2's whose sum is k. We then demonstrate a bijection between the sets $F_{n+4} \cup \bigcup_{k=1}^{n} (\{1,2,\ldots,k\} \times F_k)$ and $(\{1,2,\ldots,n+1\} \times F_{n+2}) \cup \{1,2,3\}$, which gives a bijective proof of the identity.

1 Introduction

We will interpret the k-th Fibonacci number f_k as the cardinality of the set F_k of all ordered lists of 1's and 2's that have sum k. Thus, $(f_0, f_1, f_2, f_3, f_4, f_5, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)$. For an integer m, the number mf_k will be interpreted as the cardinality of the Cartesian product $[m] \times F_k$, where $[m] := \{1, 2, 3, \ldots, m\}$.

On page 14 of *Proofs that Really Count* [1], Benjamin and Quinn mentioned that there was no known bijective proof for the identity $f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$ for $n \geq 0$. In Section 2 we define a map

$$\phi : F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k) \longrightarrow \{1,2,3\} \cup ([n+1] \times F_{n+2}),$$

and in Section 3 we define a map

$$\psi : \{1,2,3\} \cup ([n+1] \times F_{n+2}) \longrightarrow F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k).$$

A straightforward case analysis (which we omit from this paper) shows that $\psi \circ \phi$ and $\phi \circ \psi$ are both the identity, thus proving that ϕ and ψ are bijections. This provides a bijective proof of the identity $f_{n+4} + f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} + 3$ for $n \geq 0$.

The basic idea behind both directions of the bijection is to pair up a few special cases, for example, cases involving elements of $\{1, 2, 3\}$, and then to find a general way to define the rest of the bijection. For both ϕ and ψ, the general methods break into two cases, each involving adding or deleting some number of terminal 1's from a list or inserting or deleting the last 2 in a list, or both. Furthermore, the two general cases for ϕ and ψ correspond to each other: namely, the range of ϕ from Case 1 is the same as the domain for ψ in Case 1 (except for the special cases), and the range of ϕ from Case 2 is the same as the domain for ψ in Case 2.

When examples are given below, ordered lists are denoted using square braces, e.g. $[\![a_1, a_2, \ldots, a_m]\!]$. Also, Doron Zeilberger [4] has written a Maple package that implements the bijection, which may be downloaded from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/PHIL`

2 The bijection ϕ

In this section we define the bijection

$$\phi : F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k) \longrightarrow \{1, 2, 3\} \cup ([n+1] \times F_{n+2}).$$

Case 1: Consider $X \in F_{n+4}$, so X is a list of 1's and 2's that sums to $n+4$.

- If the last n numbers in X are 1's, then

$$\begin{array}{llcc}
\phi : & [\![1,1,1,1, \overbrace{1,1,\ldots,1}^{n}]\!] & \mapsto & (1, [\![\overbrace{1,1,\ldots,1}^{n+2}]\!]) \\
\phi : & [\![2,1,1, \overbrace{1,1,\ldots,1}^{n}]\!] & \mapsto & 1 \\
\phi : & [\![1,2,1, \overbrace{1,1,\ldots,1}^{n}]\!] & \mapsto & 2 \\
\phi : & [\![1,1,2, \overbrace{1,1,\ldots,1}^{n}]\!] & \mapsto & 3 \\
\phi : & [\![2,2, \overbrace{1,1,\ldots,1}^{n}]\!] & \mapsto & (1, [\![2, \overbrace{1,1,\ldots,1}^{n}]\!])
\end{array}$$

- If X ends in a string of exactly ℓ 1's, with $0 \leq \ell < n$ (so X has a 2 followed by ℓ 1's at the end) then delete the last 2 in X to get $\widehat{X}$, which is an element of F_{n+2}, and define $\phi : X \mapsto (n - \ell + 1, \widehat{X})$.

Examples for $n = 3$:

$$\begin{array}{rrcl} \phi: & [\![1,1,2,1,2]\!] & \mapsto & (4,[\![1,1,2,1]\!]) \\ \phi: & [\![1,1,2,2,1]\!] & \mapsto & (3,[\![1,1,2,1]\!]) \\ \phi: & [\![1,1,1,2,1,1]\!] & \mapsto & (2,[\![1,1,1,1,1]\!]) \end{array}$$

Case 2: Consider (i, X) where $X \in F_k$ and $i \in [k]$ (and thus $i \leq k$). Take X and append a 2 followed by $(n-k)$ 1's to get $\widetilde{X}$, which is an element of F_{n+2}, and define $\phi : (i, X) \mapsto (i, \widetilde{X})$.

Examples for $n = 3$:

$$\begin{array}{rrcl} \phi: & (1,[\![1]\!]) & \mapsto & (1,[\![1,2,1,1]\!]) \\ \phi: & (1,[\![1,1]\!]) & \mapsto & (1,[\![1,1,2,1]\!]) \\ \phi: & (2,[\![2]\!]) & \mapsto & (2,[\![2,2,1]\!]) \\ \phi: & (2,[\![2,1]\!]) & \mapsto & (2,[\![2,1,2]\!]) \end{array}$$

3 The inverse bijection ψ

In this section we define the inverse bijection

$$\psi : \{1,2,3\} \cup ([n+1] \times F_{n+2}) \longrightarrow F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k).$$

For elements of $\{1,2,3\}$ and for the element $(1, [\![\overbrace{1,1,\ldots,1}^{n+2}]\!])$ of $[n+1] \times F_{n+2}$, define ψ according to the chart below.

$$\begin{array}{rccr} \psi: & (1,[\![\overbrace{1,1,\ldots,1}^{n+2}]\!]) & \mapsto & [\![1,1,1,1,\ \overbrace{1,1,\ldots,1}^{n}]\!] \\ \psi: & 1 & \mapsto & [\![2,1,1,\ \overbrace{1,1,\ldots,1}^{n}]\!] \\ \psi: & 2 & \mapsto & [\![1,2,1,\ \overbrace{1,1,\ldots,1}^{n}]\!] \\ \psi: & 3 & \mapsto & [\![1,1,2,\ \overbrace{1,1,\ldots,1}^{n}]\!] \end{array}$$

For all cases **not** covered by the chart above, we define ψ as follows. Consider (i, Y), where $Y \in F_{n+2}$ and $i \in [n+1]$.

Case 1: If Y ends with at least $(n+1-i)$ 1's, then insert a 2 before the last $(n+1-i)$ 1's to get $\widetilde{Y}$ and define $\psi : (i, Y) \mapsto \widetilde{Y}$.

Example: $\psi : (1, [\![2, \overbrace{1,1,\ldots,1}^{n}]\!]) \mapsto [\![2,2,\overbrace{1,1,\ldots,1}^{n}]\!]$. (This case matches up with one of the special cases for ϕ.)

Examples for $n = 3$:

$$\begin{array}{rrcl} \psi: & (4, [\![1,1,1,2]\!]) & \mapsto & [\![1,1,1,2,2]\!] \\ \psi: & (3, [\![2,1,1,1]\!]) & \mapsto & [\![2,1,1,2,1]\!] \\ \psi: & (2, [\![1,1,1,1,1]\!]) & \mapsto & [\![1,1,1,2,1,1]\!] \end{array}$$

Case 2: If one of the last $n+1-i$ entries in Y is a 2, then delete the last 2 in Y and all 1's following that 2 to get $\widehat{Y}$. Define $\psi : (i, Y) \mapsto (i, \widehat{Y})$.

Examples for $n = 3$:

$$\begin{array}{rrcl} \psi: & (1, [\![1,2,1,1]\!]) & \mapsto & (1, [\![1]\!]) \\ \psi: & (1, [\![1,1,2,1]\!]) & \mapsto & (1, [\![1,1]\!]) \\ \psi: & (2, [\![2,2,1]\!]) & \mapsto & (2, [\![2]\!]) \\ \psi: & (2, [\![2,1,2]\!]) & \mapsto & (2, [\![2,1]\!]) \end{array}$$

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References

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