NAME: (print!) Kelsey Horan

E-Mail address: khoran@eden.rutgers.edu

MATH 356, Dr. Z. , Final Exam, Mon., Dec. 23, 2013, 8-11am, SEC-218

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1. 10 (out of 10)
2. 10 (out of 10)
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4. 10 (out of 10)
5. 10 (out of 10)
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7. 10 (out of 10)
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14. 10 (out of 10)
15. 8 (out of 10)
16. 10 (out of 10)
17. 10 (out of 10)
18. 10 (out of 10)
19. 10 (out of 10)
20. 10 (out of 10)

tot.: (out of 200)

195

1. Using the **formula** (no credit for other methods!), find the unique x between 0 and 98 such that

$$x \equiv 7 \pmod 9 \quad , \quad x \equiv 6 \pmod{11} \quad .$$

Reminder: The unique solution of the system $x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$ in $0 \le x < m_1 m_2$, when m_1 and m_2 are relatively prime

$$x \equiv a_1 m_2 [m_2^{-1}]_{m_1} + a_2 m_1 [m_1^{-1}]_{m_2} \pmod{m_1 m_2} \quad .$$

(Note: you may find the modular inverse by trial-and-error rather than by the 'official' way, using the Extended Euclidean Algorithm.)

Ans.: $x =$ 61

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$a_1 = 7$ $\quad$ $a_2 = 6$ $\quad$ $m_1 m_2 = 99$

$m_1 = 9$ $\quad$ $m_2 = 11$

$[m_1^{-1}]_{m_2} = [9^{-1}]_{11} = 5$ $\quad$ $[m_2^{-1}]_{m_1} = [11^{-1}]_9 = 5$

$[9 \cdot 5]_{11} = [45]_{11} = 1$ $\quad$ $[11 \cdot 5]_9 = [55]_9 = 1$

$X = [(7)(11)(5) + (6)(9)(5)] \pmod{99}$

$\quad 88 \quad + 72$

$X = 61$

$[61]_9 = 7$

$[61]_{11} = 6$

$77 \times 5 = 385$, $385 - 297 = 88$

$99 + 99 = 198$, $198 + 99 = 297$, $297 + 99$

$45 \times 6 = 270$, $270 - 198 = 72$, $72 + 88 = 160$, $160 - 99 = 61$

2. (10 pts.) Prove that if you take any 3-digit positive integer written in decimal positional notation (as usual!)

$$i_2\, i_1\, i_0 \quad , \quad (1 \le i_2 \le 9 \quad , \quad 0 \le i_1 \le 9 \quad , \quad 0 \le i_0 \le 9 \quad).$$

then the 6-digit decimal integer obtained by repeating it (for example, if you take 395 you make 395395), namely

$$i_2\, i_1\, i_0\, i_2\, i_1\, i_0 \quad ,$$

is divisible by 13.

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The divisibility test for 13 is if the alternating sum of ~~the~~ every 3 digits is divisible by 13 (or 0 mod 13) then the number is divisible by 13. $i_2 i_1 i_0 | i_2 i_1 i_0$

The alternating sum will be $i_2 i_1 i_0 - i_2 i_1 i_0 = 0$

the alternating sum is 0 mod 13 $\Rightarrow 13 | i_2 i_1 i_0 i_2 i_1 i_0$

CORRECT, BUT HERE IS A PROOF FROM SCRATCH:

$$i_2 i_1 i_0 i_2 i_1 i_0 = i_2 \cdot 10^5 + i_1 \cdot 10^4 + i_0 \cdot 10^3 + i_2 \cdot 10^2 + i_1 \cdot 10^1 + i_0 \cdot 10^0$$

$$= 10^3 (i_2 \cdot 10^2 + i_1 \cdot 10 + i_0) + 1 \cdot (i_2 \cdot 10^2 + i_1 \cdot 10 + i_0)$$

$$= (10^3 + 1)(i_2 \cdot 10^2 + i_1 \cdot 10 + i_0) =$$

$$1001 \cdot (i_2 \cdot 10^2 + i_1 \cdot 10 + i_0) = 13 [7 \cdot 11 (100 i_2 + 10 i_1 + i_0)]$$

3. (10 pts.) State Wilson's theorem, and verify it empirically for $p = 13$.

$(p-1)! \equiv -1 \pmod{p}$

$(13-1)! = 12 \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1$

If we pair the numbers with their multiplicative inverses mod 13

$[2 \cdot 6]_{13} = 1$

$[3 \cdot 9]_{13} = 1$

$[5 \cdot 8]_{13} = 1$

$[10 \cdot 4]_{13} = 1$

$[7 \cdot 11]_{13} = 1$

13
26
39

$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= (2 \cdot 6)(3 \cdot 9)(5 \cdot 8)(10 \cdot 4)(7 \cdot 11)(1 \cdot 12) \pmod{13}$

$= \cancel{4}\ 12 \bmod 13 = -1 \pmod{13}$

✓ 10

4. (10 pts.) Use the Fermat primality test to investigate whether 13 is prime or composite by picking **two** random a's between 2 and 12.

if $a^{p-1} \equiv 1 \pmod{p}$ then p is probably prime

my random a

$a = 2$: $2^{12} \stackrel{?}{\equiv} 1 \pmod{13}$ ✓

$a = 2$ is a witness to p's primality

$2^1 \equiv 2$

$2^2 \equiv 4$

$2^4 \equiv 3$

$2^8 \equiv 9$

$2^{12} = 2^8 \cdot 2^4 \equiv (9 \cdot 3) = 1$

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$a = 3$: $3^{12} \stackrel{?}{\equiv} 1 \pmod{13}$ ✓

$3^1 \equiv 3$

$3^2 \equiv 9$

$3^4 \equiv 3$

$3^8 \equiv 9$

$3^{12} = 3^8 \cdot 3^4 \equiv (9 \cdot 3) \equiv 1 \pmod{13}$

13
26
39
+13
52
+13
65
+13
78
+13
91

81
−78
3

3 is a witness to p's primality

5. (10 pts.) Compute $\phi(18000)$. Explain!

Ans.: $\phi(18000) =$ 4800

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prime decomposition of 18000

$10 \cdot 1800$

$10 \cdot 10 \cdot 180$

$10 \cdot 10 \cdot 10 \cdot 18$

$2^3 5^3 \cdot 18$

$2^4 \cdot 5^3 \cdot 9$

$\boxed{18000 = 2^4 \cdot 3^2 \cdot 5^3}$

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

the number of integers x

~~[illegible]~~ $1 \le x \le n-1$ relatively prime to n

$\gcd(x, n) = 1$.

$$\varphi(18000) = 18000\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 18000\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$$

$$= 2^4 \cdot 3 \cdot 5^2 \cdot 4 = 4800$$

$$\begin{array}{r} 16 \\ \times\ 4 \\ \hline 64 \\ \times\ 3 \\ \hline 192 \\ \times\ 25 \\ \hline 960 \\ 3840 \\ \hline 4800 \end{array}$$

ABBIE SHIEH'S SOLUTION

6. (10 pts.) Suppose Alice used RSA to send you the encrypted message c, using the public key e that you gave her. Check that this is an OK message (coprime to $n = pq$). Also check that the key is a valid key. If they are both OK, find her original message?, m.

$$p = 11 \quad , \quad q = 13 \quad , \quad e = 7 \quad , \quad c = 3 \quad .$$

Ans.: $m = 16$

10

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i) $n = p \cdot q = 11 \times 13 = 143$

$\phi(n) = \phi(143) = 143 \times (1 - \frac{1}{11}) \cdot (1 - \frac{1}{13}) = 10 \times 12 = 120$

$\gcd(e, \phi(n)) = \gcd(7, 120) = \gcd(120, 7) = 1$, so e is ok.

ii) $d = e^{-1} \bmod \phi(n) = 7^{-1} \bmod 120 = -17 + 120 = 103$

(since $1 = 120 - 17 \times 7$)

iii) $\gcd(c, n) = \gcd(3, 143) = \gcd(143, 3) = 1$, so c is ok.

iv) $m = c^d \bmod n = 3^{103} \bmod 143 = 3^{64} \cdot 3^{32} \cdot 3^{4} \cdot 3^{2} \cdot 3^{1} \pmod{143}$

$= (-62) \cdot 9 \cdot (-62) \cdot 9 \cdot 3 \pmod{143}$

$= (-17) \cdot (-62) \cdot 3 \pmod{143}$

$= 16$

$3^1 \bmod 143 = 3$

$3^2 \bmod 143 = 9$

$3^4 \bmod 143 = -62$

$3^8 \bmod 143 = -17$

$3^{16} \bmod 143 = 3$

$3^{32} \bmod 143 = 9$

$3^{64} \bmod 143 = -62$

7. (10 pts.) Prove that for every positive integer n, the number of partitions of n into odd parts equals the number of partitions of n with distinct parts.

$$\text{distinct}(n) = \prod_{i=1}^{k} (1+q^i)$$

$$\text{odd}(n) = \prod_{i=1}^{k} \frac{1}{1-q^{2i-1}}$$

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$$\prod_{i=1}^{k} (1+q^i) = (1+q)(1+q^2)(1+q^3)\ldots$$

geometric

$$= \left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^4}{1-q^2}\right)\left(\frac{1-q^6}{1-q^3}\right)\left(\frac{1-q^8}{1-q^4}\right)\left(\frac{1-q^{10}}{1-q^5}\right)$$

Only even powers of q appear in the numerator because they are of the form $2i$.

They cancel with all of the even denominators

$$\left(\frac{\cancel{1-q^2}}{1-q}\right)\left(\frac{\cancel{1-q^4}}{\cancel{1-q^2}}\right)\left(\frac{\cancel{1-q^6}}{1-q^3}\right)\left(\frac{\cancel{1-q^8}}{\cancel{1-q^4}}\right)\ldots = \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^3}\right)\left(\frac{1}{1-q^5}\right)\ldots$$

$$= \prod_{i=1}^{k}\left(\frac{1}{1-q^{2i-1}}\right)$$

thus ~~they~~ the number of partitions are equal

$$z+1 = \frac{1-z^2}{1-z}$$

$$1+z = \frac{1-z^2}{1-z}$$

8. (10 pts.) Prove that if p is prime, and $2^p - 1$ is also prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number.

$n = 2^{p-1}(2^p - 1)$

perfect number $\quad 6(n) = 2n.$

FLA

$n = 2^{p-1}(2^p - 1) \qquad 6(n) = (1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1}) \cdots (1 + p_k + \dots + p_k^{\alpha_k})$

$6(n) = (1 + (2^p - 1))(1 + 2 + 2^2 + \dots + 2^{p-1})$

$= 2^p (1 + 2 + \dots + 2^{p-1})$

$= 2^p (2^{p} - 1) \qquad 1 + 2 + \dots + 2^{p-1} = \frac{2^p - 1}{2 - 1} = 2^p - 1$

$= 2 \left[2^{p-1}(2^p - 1) \right] = 2n$

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9. (10 pts.) What is $\mu(2002)$? ($\mu(n)$ is the famous Möbius function). Explain!

Ans.: $\mu(2002) =$ 1

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$$\mu(n) = \begin{cases} 0 & \text{if prime decomp has a square} \\ (-1)^k & \text{where } k \text{ is \# of primes in the decomposition.} \end{cases}$$

$2002 = 2 \cdot 1001 = 2 \cdot 7 \cdot 11 \cdot 13$

2002 is square free so $\mu(2002) \neq 0$

$k = 4$ because $|\{2, 7, 11, 13\}| = 4$

$(-1)^4 = 1$

10. (10 pts.) Using the four rules below (most famously the Quadratic Reciprocity Law) decide whether 17 is a quadratic residue modulo 101. Explain everything.

Ans.: 17 is / is not a quadratic residue mod 101 .

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Rule 1: If p is an odd prime and a and b are not multiples of p, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\cdot\left(\frac{b}{p}\right)$$

Rule 2: If p is an odd prime then

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} .$$

Rule 3: If p is an odd prime then

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} .$$

Rule 4: (The quadratic Reciprocity Law)
If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} .$$

$$\left(\frac{17}{101}\right)\left(\frac{101}{17}\right) = (-1)^{16\cdot 100/4} = 1 \Rightarrow \left(\frac{17}{101}\right) = \left(\frac{101}{17}\right)$$

$$\left(\frac{101}{17}\right) = \left(\frac{-1}{17}\right) = (-1)^{16/2} = (-1)^{8} = 1$$

$101 \equiv -1 \pmod{17}$

$$1 = \left(\frac{-1}{17}\right) = \left(\frac{101}{17}\right) = \left(\frac{17}{101}\right) = 1 \Rightarrow$$ 17 is a quadratic Residue of 101

11. (10 pts) Apply the famous Bressoud-Zeilberger map to the pair

$$\lambda = (7,5,3,2,2,2,1,1,1) \quad , \quad j = 2 \quad .$$

Call the output (λ', j'). Then apply it to the output, and show that you get (λ, j) back.

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Ans.: $\lambda' =$ $(14,6,4,2,1,1,1)$ $\qquad j' =$ 1

Reminder: Let $\lambda = (\lambda(1), \ldots, \lambda(t))$, where t is the number of parts. If $t + 3j \geq \lambda(1)$ then $\lambda' = (t + 3j - 1, \lambda(1) - 1, \ldots, \lambda(t) - 1)$ (erasing all zeros at the end, of course), and $j' = j - 1$. Otherwise $\lambda' = (\lambda(2) + 1, \ldots, \lambda(t) + 1, 1^{\lambda(1)-3j-t-1})$, and $j' = j + 1$.

$\varphi(\lambda) = \varphi(7,5,3,2,2,2,1,1,1) \qquad j=2$

$t = 9 \qquad t+3j = 9+6 = 15 \geq \lambda_1 = 7$

$\lambda_1 = 7$

$t+3j-1 =$

$\lambda' = (14,6,4,2,1,1,1) \qquad j' = j-1 = 2-1 = 1$

$\varphi(\lambda') = \varphi^2(\lambda) = \varphi(14,6,4,2,1,1,1)$

$\lambda'_1 = 14 \qquad t' = 7 \qquad t'+3j' = 7+3 = 10 < 14 = \lambda'_1$

$\lambda = \lambda'' = (\cancel{14}\,7,5,3,2,2,2,1,1,1) \qquad j'' = j = j'+1 = 1+1 = 2.$

number of ones $\lambda_1 - 3j - t - 1 = 14 - 7 - 3 - 1 = 14 - 11 = 3$

12. (10 pts.) Apply Franklin's bijection to the distinct partition $\lambda = (15, 14, 13, 12, 6, 5, 2)$, if it is applicable. If it is indeed applicable, call the output λ', and apply Franklin's bijection to λ' and show that you get λ back.

Ans.: $\lambda' = (16, 15, 13, 12, 6, 5)$

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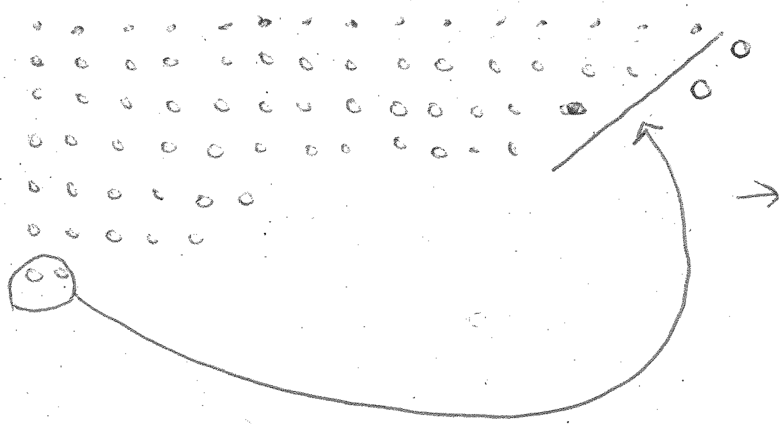
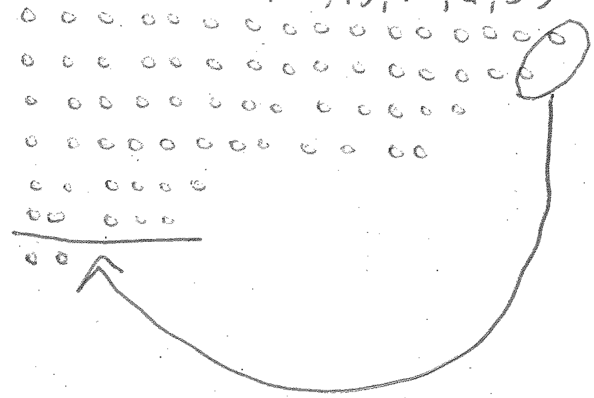
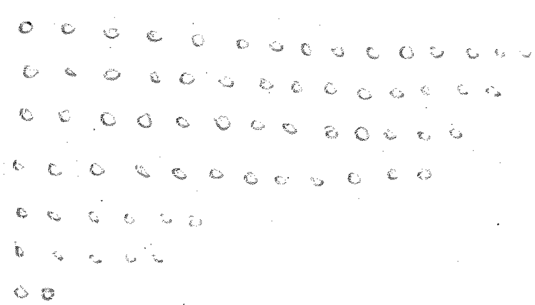
$\lambda = (15,14,13,12,6,5,2)$



$\lambda' = (16,15,13,12,6,5)$





$\lambda'' = \lambda = (15,14,13,12,6,5,2)$

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13. (10 pts.) Let n and k be a positive integers, with $k \leq n$. Prove that the number of partitions of n with exactly k parts equals the number of partitions of n whose largest part is k.

For every partition λ there is a λ' calculatable from the Ferrier Diagram of the partition illustrating this bijection.

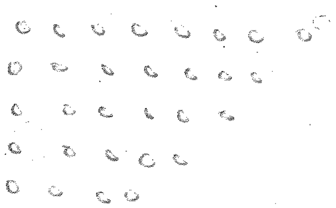
Let $n = 30$

$k = 5$

partitions of 30 with k parts example

$\lambda = (8, 7, 6, 5, 4)$

F Diag



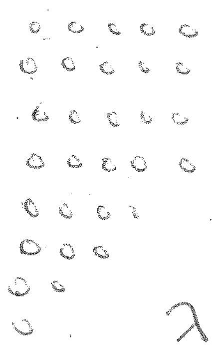
take λ' to be the colums of the F diagram of λ,

so for

$\lambda' = 5, 5, 5, 5, 4, 3, 2, 1$

this is also a partition of $n = 30$

apply the same map to get back



$\lambda'' = \lambda = (8, 7, 6, 5, 4)$

this map is a bijection.

Thus,

number of partitions of n with k parts equals number of partitions of n whose largest part is k.

14. (10 pts.) Convert $\frac{37}{55}$ into a simple continued fraction.

Ans.: $\frac{37}{55} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{18}}}$

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$$CF\left(\frac{37}{55}\right) = 0 + \frac{1}{CF\left(\frac{55}{37}\right)} = 0 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{18}}}$$

$$CF\left(\frac{55}{37}\right) = 1 + \frac{1}{CF\left(\frac{37}{18}\right)} = 1 + \cfrac{1}{2+\cfrac{1}{18}}$$

$$CF\left(\frac{37}{18}\right) = 2 + \frac{1}{CF(18)} = 2 + \frac{1}{18}$$

ABBIE SHIEH'S SOLUTION

15. (10 pts.) Evaluate the infinite continued fraction $x = [5, 1, 3, 1, 3, 1, 3, 1, 3, \ldots]$ (i.e. $x = [5, (1, 3)^\infty]$ that starts with 5 followed by 1, 3 repeated an infinite number of times) as a quadratic irrationality.

Ans.: $x = \frac{7}{2} + \frac{\sqrt{21}}{2}$ ✓ 10

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let $y = [1, 3]^\infty$.

then $y = 1 + \frac{1}{3 + \frac{1}{y}} = 1 + \frac{y}{3y+1} = \frac{4y+1}{3y+1}$

$3y^2 + y = 4y + 1$

$3y^2 - 3y - 1 = 0 \implies y = \frac{3 + \sqrt{9+12}}{2 \cdot 3} = \frac{3+\sqrt{21}}{6} = \frac{1}{2} + \frac{\sqrt{21}}{6}$

So $x = [5, y] = 5 + \frac{1}{y}$

$= 5 + \frac{1}{\frac{1}{2} + \frac{\sqrt{21}}{6}}$

$= 5 + \frac{6}{3+\sqrt{21}} = 5 + \frac{6(\sqrt{21}-3)}{21-9}$

$= 5 + \frac{\sqrt{21}-3}{2}$

$= \frac{7}{2} + \frac{\sqrt{21}}{2}$

16. (10 pts.) In Planet Z there are 9 days in the week, and the year-length is always the same (no leap years!), consisting of 400 days.
If today is 3-Day, what day of the week would it be (on planet Z) at the same date as today, but exactly 1000 years later? Explain!

Ans.: It would be 7-Day .

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$400 \pmod 9$ to calculate how much the day changes each year

$= 4 \Rightarrow +4$ to the day of the week for every year

$3 + (1000) \cdot (4) \pmod 9 \equiv 3 + 4 \pmod 9 \equiv 7$.

start day

number of years ahead

multiply because over 1000 yrs goes up 4, 1000 times

number of day goes up each year by 4

finds day of week

$$\begin{array}{r} 444 \\ 9\overline{)4000} \\ \underline{36} \\ \cancel{40} \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

R=4

17. (10 pts.) Using the Extended Euclidean algorithm (no credit for other methods!), find out whether it is possible to express 1 as a linear combination

$$1 = m \cdot 23 + n \cdot 97 \quad ,$$

for some integers m and n, and if it is possible, find m and n.

Ans.: $m = 38$ $n = -9$.

10

$\gcd(97, 23) = \gcd(23, 5) = \gcd(5, 3) = \gcd(3, 2)$
$= \gcd(2, 1) = 1$

$\Rightarrow$ linear combination possible!!

$97 = 23 \cdot 4 + 5 \longrightarrow 5 = \boxed{97} - 4\boxed{23}$

$23 = 5 \cdot 4 + 3$
$= 4 \cdot 5 + 3 \longrightarrow 3 = \boxed{23} - 4 \cdot 5 = \boxed{23} - 4(\boxed{97} - 4\boxed{23})$
$= \boxed{23} - 4\boxed{97} + 16\boxed{23}$
$= 17\boxed{23} - 4\boxed{97}$

$5 = 3 \cdot 1 + 2 \longrightarrow 2 = 5 - 3 = \boxed{97} - 4\boxed{23} - 17\boxed{23} + 4\boxed{97}$
$= 5\boxed{97} - 21\boxed{23}$

$3 = 2 \cdot 1 + 1 \longrightarrow 1 = 3 - 2 = 17\boxed{23} - 4\boxed{97} - 5\boxed{97} + 21\boxed{23}$

$1 = 38\boxed{23} - 9\boxed{97}$

18. (10 pts.) Express the integer 487 (written in our usual (base 10) notation) in base 7, in (i) sparse notation (4 pts) (ii) dense notation (3 pts) (iii) base-seven positional notation (3 pts)

Ans.: (i) $1 \cdot 7^3 + 2 \cdot 7^2 + 6 \cdot 7^1 + 4 \cdot 7^0$

(ii) $1 \cdot 7^3 + 2 \cdot 7^2 + 6 \cdot 7^1 + 4 \cdot 7^0$

(iii) 1264

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487 base10 → base7

(i) $1 \cdot 7^3 + 2 \cdot 7^2 + 6 \cdot 7^1 + 4 \cdot 7^0$

343   2(49) = 98   42   4

(ii) dense is sparse because there are no powers of 7 with zero coefficients.

19. (10 pts.) Let T_n be the Tribonacci numbers, defined by

$$T_1 = 1 \quad , \quad T_2 = 1 \quad , \quad T_3 = 1 \quad ,$$

and for $n \geq 4$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad .$$

Give a Zeilberger-style proof of the following identity,

$$T_{n+2} = 4T_{n-1} + 3T_{n-2} + 2T_{n-3} \quad .$$

by checking it empirically for $n = 4, 5, 6, 7$.

$T_4 = T_3 + T_2 + T_1 = 3$

$T_5 = T_4 + T_3 + T_2 = 5$

$T_6 = T_5 + T_4 + T_3 = 9$

$T_7 = T_6 + T_5 + T_4 = 17$

$T_8 = T_7 + T_6 + T_5 = 31$

$T_9 = T_8 + T_7 + T_6 = 48 + 9 = 57$

$n = 4$:

$T_6 = 4T_3 + 3T_2 + 2T_1$

$9 = 4(1) + 3(1) + 2(1) = 9$ ✓

$n = 5$:

$T_7 = 4T_4 + 3T_3 + 2T_2$

$17 = 4(3) + 3(1) + 2(1) = 12 + 3 + 2 = 17$ ✓

$n = 6$:

$T_8 = 4T_5 + 3T_4 + 2T_3$

$31 = 4(5) + 3(3) + 2(1) = 20 + 9 + 2 = 31$ ✓

$n = 7$:

$T_9 = 4T_6 + 3T_5 + 2T_4$

$57 = 4(9) + 3(5) + 2(3) = 36 + 15 + 6 = 57$ ✓

✓ 10

ABBIE SHIE'S SOLUTION

20. (10 pts.) Use the Fundamental Theorem of Discrete Calculus to prove the identity

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4}n(n+1)(n+2)(n+3) \quad .$$

proof:

$a(i) = i(i+1)(i+2) \qquad S(n) = \frac{1}{4}n(n+1)(n+2)(n+3)$

$S(0) = \frac{1}{4} \cdot 0(0+1)(0+2)(0+3) = 0$

$$\begin{aligned} S(n) - S(n-1) &= \frac{1}{4}n(n+1)(n+2)(n+3) - \frac{1}{4}(n-1)(n-1+1)(n-1+2)(n-1+3) \\ &= \frac{1}{4}n(n+1)(n+2)(n+3) - \frac{1}{4}(n-1)(n)\cdot(n+1)(n+2) \\ &= \frac{1}{4}n(n+1)(n+2)[n+3-(n-1)] \\ &= \frac{1}{4}n(n+1)(n+2)\cdot 4 \\ &= n(n+1)(n+2) = a(n) \end{aligned}$$

Thus $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

10