1. Using the **formula** (no credit for other methods!), find the unique x between 0 and 98 such that

$$x \equiv 7 \pmod 9 \quad , \quad x \equiv 6 \pmod{11} \quad .$$

Reminder: The unique solution of the system $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}$ in $0 \le x < m_1 m_2$, when m_1 and m_2 are relatively prime

$$x \equiv a_1 m_2 [m_2^{-1}]_{m_1} + a_2 m_1 [m_1^{-1}]_{m_2} \pmod{m_1 m_2} \quad .$$

(Note: you may find the modular inverse by trial-and-error rather than by the 'official' way, using the Extended Euclidean Algorithm.)

2. (10 pts.) Prove that if you take any 3-digit positive integer written in decimal positional notation (as usual!)

$$i_2\, i_1\, i_0 \quad , \quad (1 \le i_2 \le 9 \quad , \quad 0 \le i_1 \le 9 \quad , \quad 0 \le i_0 \le 9 \quad).$$

then the 6-digit decimal integer obtained by repeating it (for example, if you take 395 you make 395395), namely

$$i_2\, i_1\, i_0\, i_2\, i_1\, i_0 \quad ,$$

is divisible by 13.

3. (10 pts.) State Wilson's theorem, and verify it empirically for $p = 13$.

4. (10 pts.) Use the Fermat primality test to investigate whether 13 is prime or composite by picking **two** random a's between 2 and 12

5. (10 pts.) Compute $\phi(18000)$. Explain!

6. (10 pts.) Suppose Alice used RSA to send you the encrypted message c, using the public key e that you gave her. Check that this is an OK message (coprime to $n = pq$). Also check that the key is a valid key. If they are both OK, find her original message?, m.

$$p = 11 \quad , \quad q = 13 \quad , \quad e = 7 \quad , \quad c = 3 \quad .$$

7. (10 pts.) Prove that for every positive integer n, the number of partitions of n into odd parts equals the number of partitions of n with distinct parts.

8. (10 pts.) Prove that if p is prime, and $2^p - 1$ is also prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number.

9. (10 pts.) What is $\mu(2002)$? ($\mu(n)$ is the famous Möbius function).

10. (10 pts.) Using the four rules below (most famously the Quadratic Reciprocity Law) decide whether 17 is a quadratic residue modulo 101. Explain everything.

Rule 1: If p is an odd prime and a and b are not multiples of p, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$$

Rule 2: If p is an odd prime then

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \quad .$$

Rule 3: If p is an odd prime then

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \quad .$$

Rule 4: (The quadratic Reciprocity Law)
If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} \quad .$$

11. (10 pts) Apply the famous Bressoud-Zeilberger map to the pair

$$\lambda = (7,5,3,2,2,2,1,1,1) \quad , \quad j = 2 \quad .$$

Call the output (λ', j'). Then apply it to the output, and show that you get (λ, j) back.
Reminder: Let $\lambda = (\lambda(1), \dots, \lambda(t))$, where t is the number of parts. If $t + 3j \geq \lambda(1)$ then $\lambda' = (t + 3j - 1, \lambda(1) - 1, \dots, \lambda(t) - 1)$ (erasing all zeros at the end, of course), and $j' = j - 1$. Otherwise $\lambda' = (\lambda(2) + 1, \dots, \lambda(t) + 1, 1^{\lambda(1)-3j-t-1})$, and $j' = j + 1$.
12. (10 pts.) Apply Franklin's bijection to the distinct partition $\lambda = (15, 14, 13, 12, 6, 5, 2)$, if it is applicable. If it is indeed applicable, call the output λ', and apply Franklin's bijection to λ' and show that you get λ back.
13. (10 pts.) Let n and k be a positive integers, with $k \leq n$. Prove that the number of partitions of n with exactly k parts equals the number of partitions of n whose largest part is k.
14. (10 pts.) Convert $\frac{37}{55}$ into a simple continued fraction.
15. (10 pts.) Evaluate the infinite continued fraction $x = [5, 1, 3, 1, 3, 1, 3, 1, 3, \dots]$ (i.e. $x = [5, (1, 3)^\infty]$ that starts with 5 followed by $1, 3$ repeated an infinite number of times) as a quadratic irrationality.
16. (10 pts.) In Planet Z there are 9 days in the week, and the year-length is always the same (no leap years!), consisting of 400 days.
If today is 3-Day, what day of the week would it be (on planet Z) at the same date as today, but exactly 1000 years later? Explain!
17. (10 pts.) Using the Extended Euclidean algorithm (no credit for other methods!), find out whether it is possible to express 1 as a linear combination

$$1 = m \cdot 23 + n \cdot 97 \quad ,$$

for some integers m and n, and if it is possible, find m and n.
18. (10 pts.) Express the integer 487 (written in our usual (base 10) notation) in base 7, in (i) sparse notation (4 pts) (ii) dense notation (3 pts) (iii) base-seven positional notation (3 pts)
19. (10 pts.) Let T_n be the Tribonacci numbers, defined by

$$T_1 = 1 \quad , \quad T_2 = 1 \quad , \quad T_3 = 1 \quad ,$$

and for $n \geq 4$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad .$$

Give a Zeilberger-style proof of the following identity,

$$T_{n+2} = 4T_{n-1} + 3T_{n-2} + 2T_{n-3} \quad .$$

by checking it empirically for $n = 4, 5, 6, 7$.

20. (10 pts.) Use the Fundamental Theorem of Discrete Calculus to prove the identity

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4} n(n+1)(n+2)(n+3) \quad .$$