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MATH 356, Dr. Z. , Make Up Final Exam, Mon., Jan. 27, 2025, 3:00-6:00pm, Hill 704

Do not write below this line (office use only)

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1. (out of 10)
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18. (out of 10)
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20. (out of 10)

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tot.: (out of 200)

1. Find the unique x between 0 and 1019 such that

$$x \equiv 5 \pmod{20} \quad , \quad x \equiv 5 \pmod{51} \quad .$$

Ans.: $x =$

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2. (10 pts.) Prove that if you take any 3-digit positive integer written in decimal positional notation (as usual!)

$$i_2\, i_1\, i_0 \quad , \quad (1 \leq i_2 \leq 9 \quad , \quad 0 \leq i_1 \leq 9 \quad , \quad 0 \leq i_0 \leq 9 \quad).$$

then the 6-digit decimal integer obtained by repeating it (for example, if you take 395 you make 395395), namely

$$i_2\, i_1\, i_0\, i_2\, i_1\, i_0 \quad ,$$

is divisible by 7.

3. (10 pts.) State Wilson's theorem, and verify it empirically for $p = 17$.

4. (10 pts.) Use the Fermat primality test to investigate whether 13 is prime or composite by picking **two** random a's between 2 and 12

5. (10 pts.) Compute $\phi(10^{10})$. You can leave the answer in facored form. Explain!

Ans.: $\phi(10^{10}) =$

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6. (10 pts.) Suppose Alice used RSA to send you the encrypted message c, using the public key e that you gave her. Check that this is an OK message (coprime to $n = pq$). Also check that the key is a valid key. If they are both OK, find her original message?, m.

$$p = 11 \quad , \quad q = 13 \quad , \quad e = 7 \quad , \quad c = 3 \quad .$$

Ans.: $m =$

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7. (10 pts.) Prove that for every positive integers n and k, the number of partitions of n with exactly k parts equals the number of partitions of n with largest part k.

8. (10 pts.) State and prove Fermat's Little Theorem.

KA **9**. (10 pts.) What is $\mu(2002)$? ($\mu(n)$ is the famous Möbius function).

Ans.: $\mu(2002) =$

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10. (10 pts.) Decide whether 121 is a quadratic residue modulo 1001. Explain.

Ans.: 121 is/ is not a quadratic residue mod 1001

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11. (10 pts) Apply the famous Bressoud-Zeilberger map to the pair

$$\lambda = (7,5,3,2,2,2,2,1,1,1) \quad , \quad j = 2 \quad .$$

Call the output (λ', j'). Then apply it to the output, and show that you get (λ, j) back.

Ans.: $\lambda' =$ $j' =$

Reminder: Let $\lambda = (\lambda(1), \ldots, \lambda(t))$, where t is the number of parts. If $t + 3j \geq \lambda(1)$ then $\lambda' = (t + 3j - 1, \lambda(1) - 1, \ldots, \lambda(t) - 1)$ (erasing all zeros at the end, of course), and $j' = j - 1$. Otherwise $\lambda' = (\lambda(2) + 1, \ldots, \lambda(t) + 1, 1^{\lambda(1) - 3j - t - 1})$, and $j' = j + 1$.

12. (10 pts.) Apply Franklin's bijection to the distinct partition $\lambda = (15, 14, 13)$, if it is applicable. If it is indeed applicable, call the output λ', and apply Franklin's bijection to λ' and show that you get λ back.

Ans.: $\lambda' =$

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13. (10 pts.) What is the day of the week on Jan. 27, 9025 . Explain!

14. (10 pts.) Convert $\frac{11}{77}$ into a simple continued fraction.

Ans.: $\frac{11}{77} =$

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15. (10 pts.) Evaluate the infinite continued fraction $x = [5, 1, 1, 1, 1, , \ldots]$ (i.e. $x = [5, (1)^{\infty}]$ that starts with 5 followed by 1 repeated an infinite number of times) as a quadratic irrationality.

Ans.: $x =$

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16. (10 pts.) In Planet Z there are 10 days in the week, and the year-length is always the same (no leap years!), consisting of 100 days.
If today is 3-Day, what day of the week would it be (on planet Z) at the same date as today, but exactly 1000 years later? Explain!

Ans.: It would be -Day .

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17. (10 pts.) Using the Extended Euclidean algorithm (no credit for other methods!), find out whether it is possible to express 1 as a linear combination

$$1 = m \cdot 13 + n \cdot 17 \quad ,$$

for some integers m and n, and if it is possible, find m and n.

Ans.: $m =$ $n =$.

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18. (10 pts.) Express the integer 32 (written in our usual (base 10) notation) in base 3, in (i) sparse notation (4 pts) (ii) dense notation (3 pts) (iii) base-seven positional notation (3 pts)

Ans.: (i)

(ii)

(iii)

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19. (10 pts.) Let T_n be the Tribonacci numbers, defined by

$$T_1 = 1 \quad , \quad T_2 = 1 \quad , \quad T_3 = 1 \quad ,$$

and for $n \geq 4$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad .$$

Give a Zeilberger-style proof of the following identity,

$$T_{n+2} = 4T_{n-1} + 3T_{n-2} + 2T_{n-3} \quad .$$

by checking it empirically for $n = 4, 5, 6, 7$.

20. (10 pts.) Use the Fundamental Theorem of Discrete Calculus to prove the identity

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad .$$