NAME: (print!) ____

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MATH 356, Dr. Z. , Final Exam, Mon., Dec. 16, 2024, 8:00-11:00am, SEC-204

Do not write below this line (office use only)

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tot.: (out of 200)

1. Using the **formula** (no credit for other methods!), find the unique x between 0 and 98 such that

$$x \equiv 7 \pmod{9} \quad , \quad x \equiv 6 \pmod{11} \quad .$$

Reminder: The unique solution of the system $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}$ in $0 \leq x < m_1 m_2$, when m_1 and m_2 are relatively prime

$$x \equiv a_1 m_2 [m_2^{-1}]_{m_1} + a_2 m_1 [m_1^{-1}]_{m_2} \pmod{m_1 m_2} \quad .$$

(Note: you may find the modular inverse by trial-and-error rather than by the 'official' way, using the Extended Euclidean Algorithm.)

Ans.: $x =$

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2. (10 pts.) Prove that if you take any 3-digit positive integer written in decimal positional notation (as usual!)

$$i_2\, i_1\, i_0 \quad , \quad (1 \leq i_2 \leq 9 \quad , \quad 0 \leq i_1 \leq 9 \quad , \quad 0 \leq i_0 \leq 9 \quad).$$

then the 6-digit decimal integer obtained by repeating it (for example, if you take 395 you make 395395), namely

$$i_2\, i_1\, i_0\, i_2\, i_1\, i_0 \quad ,$$

is divisible by 13.

3. (10 pts.) State Wilson's theorem, and verify it empirically for $p = 13$.

4. (10 pts.) Use the Fermat primality test to investigate whether 13 is prime or composite by picking **two** random a's between 2 and 12

5. (10 pts.) Compute $\phi(18000)$. Explain!

Ans.: $\phi(18000) =$

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6. (10 pts.) Suppose Alice used RSA to send you the encrypted message c, using the public key e that you gave her. Check that this is an OK message (coprime to $n = pq$). Also check that the key is a valid key. If they are both OK, find her original message?, m.

$$p = 11 \quad , \quad q = 13 \quad , \quad e = 7 \quad , \quad c = 3 \quad .$$

Ans.: $m =$

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7. (10 pts.) Prove that for every positive integer n, the number of partitions of n into odd parts equals the number of partitions of n with distinct parts.

8. (10 pts.) Prove that if p is prime, and $2^p - 1$ is also prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number.

9. (10 pts.) What is $\mu(2002)$? ($\mu(n)$ is the famous Möbius function).

Ans.: $\mu(2002) =$

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10. (10 pts.) Using the four rules below (most famously the Quadratic Reciprocity Law) decide whether 17 is a quadratic residue modulo 101. Explain everything.

Ans.: 17 is/ is not a quadratic residue mod 101

Rule 1: If p is an odd prime and a and b are not multiples of p, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$$

Rule 2: If p is an odd prime then

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \quad .$$

Rule 3: If p is an odd prime then

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \quad .$$

Rule 4: (The quadratic Reciprocity Law)
If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} \quad .$$

11. (10 pts) Apply the famous Bressoud-Zeilberger map to the pair

$$\lambda = (7, 5, 3, 2, 2, 2, 2, 1, 1, 1) \quad , \quad j = 2 \quad .$$

Call the output (λ', j'). Then apply it to the output, and show that you get (λ, j) back.

Ans.: $\lambda' =$ $j' =$

Reminder: Let $\lambda = (\lambda(1), \dots, \lambda(t))$, where t is the number of parts. If $t + 3j \geq \lambda(1)$ then $\lambda' = (t + 3j - 1, \lambda(1) - 1, \dots, \lambda(t) - 1)$ (erasing all zeros at the end, of course), and $j' = j - 1$. Otherwise $\lambda' = (\lambda(2) + 1, \dots, \lambda(t) + 1, 1^{\lambda(1)-3j-t-1})$, and $j' = j + 1$.

12. (10 pts.) Apply Franklin's bijection to the distinct partition $\lambda = (15, 14, 13, 12, 6, 5, 2)$, if it is applicable. If it is indeed applicable, call the output λ', and apply Franklin's bijection to λ' and show that you get λ back.

Ans.: $\lambda' =$

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13. (10 pts.) Let n and k be a positive integers, with $k \leq n$. Prove that the number of partitions of n with exactly k parts equals the number of partitions of n whose largest part is k.

14. (10 pts.) Convert $\frac{37}{55}$ into a simple continued fraction.

Ans.: $\frac{37}{55} =$

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15. (10 pts.) Evaluate the infinite continued fraction $x = [5, 1, 3, 1, 3, 1, 3, 1, 3, \ldots]$ (i.e. $x = [5, (1, 3)^\infty]$ that starts with 5 followed by $1, 3$ repeated an infinite number of times) as a quadratic irrationality.

Ans.: $x =$

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16. (10 pts.) In Planet Z there are 9 days in the week, and the year-length is always the same (no leap years!), consisting of 400 days.
If today is 3-Day, what day of the week would it be (on planet Z) at the same date as today, but exactly 1000 years later? Explain!

Ans.: It would be   -Day .

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17. (10 pts.) Using the Extended Euclidean algorithm (no credit for other methods!), find out whether it is possible to express 1 as a linear combination

$$1 = m \cdot 23 + n \cdot 97 \quad ,$$

for some integers m and n, and if it is possible, find m and n.

Ans.: $m =$ $\qquad$ $n =$ $\qquad$.

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18. (10 pts.) Express the integer 487 (written in our usual (base 10) notation) in base 7, in (i) sparse notation (4 pts) (ii) dense notation (3 pts) (iii) base-seven positional notation (3 pts)

Ans.: (i)

(ii)

(iii)

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19. (10 pts.) Let T_n be the Tribonacci numbers, defined by

$$T_1 = 1 \quad , \quad T_2 = 1 \quad , \quad T_3 = 1 \quad ,$$

and for $n \geq 4$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad .$$

Give a Zeilberger-style proof of the following identity,

$$T_{n+2} = 4T_{n-1} + 3T_{n-2} + 2T_{n-3} \quad .$$

by checking it empirically for $n = 4, 5, 6, 7$.

20. (10 pts.) Use the Fundamental Theorem of Discrete Calculus to prove the identity

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{1}{4} n(n+1)(n+2)(n+3) \quad .$$