Solutions to Quiz # 9 for Dr. Z.'s Number Theory Course for Nov. 21, 2013

1. (5 points) State the Möbius inversion formula, and check it empirically for all $n \in Div(30)$

sol. to 1: If $f(n)$ is any function defined on the integers, and you define a brand-new function

$$g(n) := \sum_{d|n} f(d) \quad ,$$

then you can get the function $f(n)$ back from the function $g(n)$ in terms of the formula

$$f(n) := \sum_{d|n} \mu(d) g(n/d) \quad ,$$

where $\mu(d)$ is the famous Möbius function.

Now $30 = 2 \cdot 3 \cdot 5$ so

$$Div(30) = \{1,2\} \cdot \{1,3\} \cdot \{1,5\} = \{1,2,3,6\} \cdot \{1,5\} = \{1,2,3,6,5,10,15,30\}$$

$$= \{1,2,3,6,5,10,15,30\} = \{1,2,3,5,6,10,15,30\} \quad .$$

We have for $n \in Div(30)$:

$$g(1) = f(1) \quad ,$$

$$g(2) = f(1) + f(2) \quad ,$$

$$g(3) = f(1) + f(3) \quad ,$$

$$g(5) = f(1) + f(5) \quad ,$$

$$g(6) = f(1) + f(2) + f(3) + f(6) \quad ,$$

$$g(10) = f(1) + f(2) + f(5) + f(10) \quad ,$$

$$g(15) = f(1) + f(3) + f(5) + f(15) \quad ,$$

$$g(30) = f(1) + f(2) + f(3) + f(5) + f(6) + f(10) + f(15) + f(30) \quad .$$

Now let's compute $\sum_{d|n} \mu(d) g(n/d)$ for all $n \in Div(30)$.

n=1:

$$\sum_{d|1} \mu(d) g(1/d) = \mu(1) g(1) = f(1)$$

n=2:

$$\sum_{d|2} \mu(d) g(2/d) = \mu(1) g(2) + \mu(2) g(1) = g(2) - g(1) = f(1) + f(2) - f(1) = f(2) \quad .$$

n=3:

$$\sum_{d|3} \mu(d) g(3/d) = \mu(1)g(3) + \mu(3)g(1) = g(3) - g(1) = f(1) + f(3) - f(1) = f(3) \quad .$$

n=5:

$$\sum_{d|5} \mu(d) g(5/d) = \mu(1)g(5) + \mu(5)g(1) = g(5) - g(1) = f(1) + f(5) - f(1) = f(5) \quad .$$

n=6:

$$\sum_{d|6} \mu(d) g(6/d) = \mu(1)g(6) + \mu(2)g(3) + \mu(3)g(2) + \mu(6)g(1) = g(6) - g(3) - g(2) + g(1) =$$

$$f(1) + f(2) + f(3) + f(6) - (f(1) + f(3)) - (f(1) + f(2)) + f(1) = f(6)$$

n=10:

$$\sum_{d|10} \mu(d) g(10/d) = \mu(1)g(10) + \mu(2)g(5) + \mu(5)g(2) + \mu(10)g(1) = g(10) - g(5) - g(2) + g(1) =$$

$$f(1) + f(2) + f(5) + f(10) - (f(1) + f(5)) - (f(1) + f(2)) + f(1) = f(10)$$

n=15:

$$\sum_{d|15} \mu(d) g(15/d) = \mu(1)g(15) + \mu(3)g(5) + \mu(5)g(3) + \mu(15)g(1) = g(15) - g(5) - g(3) + g(1) =$$

$$f(1) + f(3) + f(5) + f(15) - (f(1) + f(5)) - (f(1) + f(3)) + f(1) = f(15)$$

n=30:

$$\sum_{d|30} \mu(d) g(30/d)$$

$$= \mu(1)g(30) + \mu(2)g(15) + \mu(3)g(10) + \mu(5)g(6) + \mu(6)g(5) + \mu(10)g(3) + \mu(15)g(2) + \mu(30)g(1)$$

$$= g(30) - g(15) - g(10) - g(6) + g(5) + g(3) + g(2) - g(1)$$

$$f(1) + f(2) + f(3) + f(5) + f(6) + f(10) + f(15) + f(30)$$

$$-(f(1) + f(3) + f(5) + f(15)) - (f(1) + f(2) + +f(5) + f(10)) - (f(1) + f(2) + f(3) + f(6))$$

$$+(f(1) + f(5)) + (f(1) + f(3)) + (f(1) + f(2))$$

$$-f(1) = f(30) \quad .$$

2. (5 points) Using the important test (involving modular exponentiation) (no credit for brute force!) find

$\left(\frac{8}{19}\right)$

Sol. of 2.

We have to find

$$8^{(19-1)/2} \bmod 19 \quad = 8^9 \bmod 19 \quad .$$

$$8^1 \bmod 19 = 8$$

$$8^2 \bmod 19 = 64 \bmod 19 = 7$$

$$8^4 \bmod 19 = 7^2 \bmod 19 = 49 \bmod 19 = -8$$

$$8^8 \bmod 19 = (-8)^2 \bmod 19 = 7$$

So

$$8^9 \bmod 19 = 8 \cdot 8^8 \bmod 19 = 8 \cdot 7 \bmod 19 = 56 \bmod 19 = -1 \quad .$$

Ans. to 2: $\left(\frac{8}{19}\right) = -1$

Note: Some people used the rule $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ To first do

$$\left(\frac{8}{19}\right) = \left(\frac{2}{19}\right)^3 \quad ,$$

and then found out that $2^9 \bmod 19 = -1$. That's OK too.