**Definition:** A *prime number* is a positive integer (larger than 1) that is only divisible by 1 and itself.

**How to decide whether a positive integer \( n \) is prime?** (The VERY STUPID WAY).

Starting with 2, try to divide it by any integer smaller than \( n \), and see whether you ever get remainder 0. If you do, then the candidate integer \( n \) is *composite*, otherwise it is *prime*.

**Problem 5.1:** Decide whether 17 is prime using the very stupid way.

**Solution to 5.1:** \( 17/2 = 8(1), 17/3 = 5(2), 17/4 = 4(1), 17/5 = 3(2), 17/6 = 2(5), 17/7 = 2(3), 17/8 = 2(1), 17/9 = 1(8), 17/10 = 1(7), 17/11 = 1(6), 17/12 = 1(5), 17/13 = 1(4), 17/14 = 1(3), 17/15 = 1(2), 17/16 = 1(1) .

So if you divide 17 by all integers from 2 to 16 you never get 0 remainder. Hence 17 is prime.

**How to decide whether a positive integer \( n \) is prime?** (The STUPID WAY).

Since if \( n = ab \), either \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \) (why?), it is enough to check every integer \( \leq \sqrt{n} \).

**Problem 5.1':** Decide whether 17 is prime using the stupid way.

**Solution to 5.1':** \( 4 < \sqrt{17} < 5 \), so we only have to check \( 17/2 = 8(1), 17/3 = 5(2), 17/4 = 4(1) .

So if you divide 17 by all integers from 2 to \( \lfloor \sqrt{17} \rfloor = 4 \) you never get 0 remainder. Hence 17 is prime.

**How to decide whether a positive integer \( n \) is prime?** (The OK WAY).

If \( n \) is divisible by some integer \( < \sqrt{n} \), it must be divisible by some *prime* \( < \sqrt{n} \). So it is enough to check every prime \( \leq \sqrt{n} \).

**Problem 5.1'':** Decide whether 17 is prime using the OK way.

**Solution to 5.1'':** \( 4 < \sqrt{17} < 5 \), so we only have to check \( 17/2 = 8(1), 17/3 = 5(2) .

So if you divide 17 by all primes from 2 to \( \lfloor \sqrt{17} \rfloor = 4 \) you never get 0 remainder. Hence 17 is prime.

There is only one catch, how do we find out all the primes \( \leq n \). Using the OK way, we do it *recursively*, one-by-one, by using the sieve of Eratosthenes.

**Input:** A positive integer \( n \)
Output: The list of all prime numbers \( \leq n \), written in increasing order.

**Step 1:** Write down *all* the integers from 2 to \( n \)

**Step 2.0:** Cross out the (proper) multiples of 2. Look at the smallest new survivor (it happens to be 3).

**Step 2.1:** Cross out the proper multiples of 3. Look at the smallest new survivor (it happens to be 5).

**Step 2.:** Until you reach \( \sqrt{n} \), keep crossing-out the multiples of the new smallest survivor (that has not been used before).

The list of *survivors* (those that have not been crossed out), is the list of primes \( \leq n \).

**Problem 5.2:** Find all the prime numbers \( \leq 20 \).

**Solution to 5.2:**

**Step 1:**

\[
2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
\]

**Step 2.1:** Cross-out, all multiples of 2 (except 2)

\[
2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
\]

**Step 2.2:** Cross-out, all multiples of 3 (except 3)

\[
2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
\]

The smallest survivor 5 is larger than \( \sqrt{20} \), so we are done!

**Ans. to 5.2:** The list of prime numbers \( \leq 20 \) are

\[
2, 3, 7, 11, 13, 17, 19\,.
\]

**Euclid’s Proof that there are “infinitely” many primes**

Suppose that there are only finitely many primes, \( n \) of them, let’s call them, in order

\[
p_1, p_2, \ldots, p_n\,.
\]

Consider

\[
P = p_1 p_2 \cdots p_n + 1\,.
\]
This number leaves remainder 1 when divided by each of \( p_1, \ldots, p_n \), hence is either prime (larger than \( p_n \)), or is divisible by a prime larger than \( p_n \), contradiction. Hence there is always an infinite supply of prime numbers.

This can be used to construct, an *infinite* sequence of prime numbers.

Let \( p_1 = 2 \), and let \( p_n \) be the smallest prime-divisor of \( p_1 p_2 \cdots p_{n-1} + 1 \).

It starts like this: 2, 3, 7, 43, 13, . . ., and it is called the **Euclid-Mullin** sequence.