Dr. Z.'s Number Theory Lecture 24 Handout: Continued Fractions

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Important Definition: A **general (finite) continued fraction** (of length k) looks as follows

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\dots}}} \quad .$$

where it goes down k levels, and the $a_0, b_1, a_2, \dots$ are **positive integers**.

In order to save space we write it as $CF[a_0; b_1, a_1; b_2, a_2; \dots; b_k, a_k]$.

Problem 24.1: Evaluate the general continued fraction

$$2 + \frac{3}{2 + \frac{5}{4}} \quad .$$

Sol. to 24.1: We go **bottom up**

$$2 + \frac{3}{2 + \frac{5}{4}} = 2 + \frac{3}{\frac{11}{4}} = 2 + \frac{12}{11} = \frac{34}{11} \quad .$$

Ans. to 24.1: $\frac{34}{11}$.

If all the numerators, b_i are 1, then we have a **simple continued fraction** (the best kind).

Important Definition: A **simple (finite) continued fraction** (of length k) looks as follows

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} \quad .$$

where it goes down k levels. Here $a_0 \geq 0$ and $a_1, a_2, a_3 \dots > 0$.

If $k = \infty$, i.e. it goes for ever, then we have an **infinite continued fraction**.

Of course, every finite continued fraction (both simple and general) is automatically a positive **rational number**, but the converse is also true.

Important Fact: Every positive rational number a/b (where a and b are positive integers) can be written as a simple continued fraction.

Recursive algorithm:

Input: a rational number x.

Output: The continued fraction, $CF(x)$.

Write $x = \lfloor x \rfloor + \{x\}$ where $\lfloor x \rfloor$ is the **integer part** and $\{x\}$ is the **fraction part**.

If $\{x\} = 0$ (i.e. of x is an integer) RETURN x.

Otherwise

$$CF(x) = \lfloor x \rfloor + CF(1/\{x\}) \quad .$$

Problem 24.2: Convert $\frac{19}{4}$ into a continued fraction .

Sol. to 24.2: $\frac{19}{4} = 4 + \frac{3}{4}$, so

$$CF(\frac{19}{4}) = 4 + \frac{1}{CF(\frac{4}{3})} \quad .$$

Now $\frac{4}{3} = 1 + \frac{1}{3}$, so

$$CF(\frac{4}{3}) = 1 + \frac{1}{CF(3)} \quad .$$

Now $3 = 3 + 0$, so $CF(3) = 3$. Now let's go to the **back** journey

$$CF(\frac{4}{3}) = 1 + \frac{1}{3} \quad .$$

$$CF(\frac{19}{4}) = 4 + \frac{1}{1 + \frac{1}{3}} \quad .$$

Ans. to 24.2: The continued fraction representation of $\frac{19}{4}$ is $4 + \frac{1}{1+\frac{1}{3}}$, or $[4, 1, 3]$ for short.

It follows that an **infinite** (simple!) continued fraction is automatically an **irrational number**.

Important Definition: A **quadratic irrationality** is a number x that satisfies a quadratic equation

$$ax^2 + bx + c = 0 \quad ,$$

where a, b, and c are are **integers**. Equivalently, it is a number that can be written as $r + s\sqrt{Q}$ where r and s are rational numbers and Q is a positive integer that is not a perfect square.

Example: $\sqrt{2}$ is a quadratic irrationality, since it is a solution of

$$x^2 - 2 = 0 \quad .$$

(i.e. $a = 1, b = 0, c = -2$.

Note: $x = \frac{3}{2}$ satisfies the quadratic equation

$$3x^2 - 5x + 2 = 0 \quad ,$$

but it is **not** a quadratic irrationality, since it is *rational* number.

A **pure periodic** infinite continued fraction is anything of the form (let's assume that its value is > 1)

$$[a_1, \ldots, a_k, a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots] \quad ,$$

where the entries $a_1, \ldots, a_k$ repeat for ever.

Important Theorem: Every pure periodic continued fraction is a quadratic irrationality.

Proof and Algorithm: Since it is an *infinite* continued fraction, it must be irrational. Let's call

$$x = [a_1, \ldots, a_k, a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots] \quad ,$$

Then, of course

$$x = [a_1, \ldots, a_k, x] \quad ,$$

Now, evaluating $[a_1, \ldots, a_k, x]$ yields a **linear fractional** expression

$$\frac{A + Bx}{C + Dx} \quad ,$$

for some positive integers A, B, C, D. So we have the equation

$$x = \frac{A + Bx}{C + Dx} \quad .$$

Doing the algebra, gives a quadratic equation for x.

Problem 24.3: Express as a quadratic irrationality the following periodic simple continued fraction x:

$$x = [2, 3, 2, 3, 2, 3, 2, 3, \ldots \quad] \quad ,$$

where $2, 3$ repeat for ever.

Sol. to 24.3:

$$[2, 3, x] = 2 + \frac{1}{3 + \frac{1}{x}} = 2 + \frac{1}{\frac{3x+1}{x}} = 2 + \frac{x}{3x + 1} = \frac{7x + 2}{3x + 1} \quad .$$

So x satisfies

$$x = \frac{7x + 2}{3x + 1} \quad .$$

Multiplying by the denominator $3x + 1$, we get:

$$x(3x + 1) = 7x + 2 \quad ,$$

expanding:

$$3x^2 + x = 7x + 2 \quad .$$

$$3x^2 + x - 7x - 2 = 0 \quad .$$

So:

$$3x^2 - 6x - 2 = 0 \quad .$$

Solving it we get

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2 \cdot 3} = \frac{6 \pm \sqrt{60}}{6} = 1 \pm \frac{\sqrt{15}}{3} \quad .$$

But, x is positive, so it is $x = 1 + \frac{1}{3}\sqrt{15}$.

Ans. to 24.3: The continued fraction $[2,3]^*$ equals $1 + \frac{1}{3}\sqrt{15}$.

Important Definition: An *ultimately periodic* infinite continued fraction is something of the form

$$[A_1, A_2, A_3, , A_m, a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k \dots] \quad ,$$

where $[A_1, A_2, \dots, A_m]$ is the **beginning** and $a_1, \dots, a_k$ is the *ultimate period*, that repeats for ever.

Important Theorem: Every ultimately periodic continued fraction is a quadratic irrationality.

Proof/Algorithm: Let

$$x = [A_1, A_2, A_3, , A_m, a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k \dots] \quad ,$$

and

$$y = [a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k \dots] \quad .$$

We already know that y is a quadratic irrationality (and how to find it), so

$$y = A + \sqrt{B} \quad ,$$

for some rational numbers A and B. Now

$$x = [A_1, A_2, A_3, , A_m, y] \quad .$$

This means, spelling out the (finite) continued fraction

$$x = \frac{Cy + D}{Ey + F} \quad ,$$

for some integers C, D, E, F. Plugging-in y gives the answer.

Problem 24.4: Find a representation in the form $a + b\sqrt{Q}$ for rational numbers a and b and positive integer Q, for the following infinite, ultimately periodic, continued fraction x.

$$x = [1, 3, 2, 3, 2, 3, 2, 3, 2, 3, \dots \quad] \quad ,$$

where $2, 3$ repeat for ever.

Solution to 24.4: Let $y = [2, 3, 2, 3, 2, 3, 2, 3]^*$. We know from 24.3 that it equals $1 + \frac{1}{3}\sqrt{15}$.

Now

$$x = [1,3,y] = 1 + \frac{1}{3+\frac{1}{y}} = 1 + \frac{1}{\frac{3y+1}{y}} = 1 + \frac{y}{3y+1} = \frac{4y+1}{3y+1} \quad .$$

So

$$x = \frac{4(1+\frac{1}{3}\sqrt{15})+1}{3(1+\frac{1}{3}\sqrt{15})+1}$$
$$= \frac{5+\frac{4}{3}\sqrt{15}}{4+\sqrt{15}}$$

Multiplying top and bottom by the **conjugate** of the bottom, $4-\sqrt{15}$, gives

$$x = \frac{(5+\frac{4}{3}\sqrt{15})(4-\sqrt{15})}{(4+\sqrt{15})(4-\sqrt{15})}$$

$$= \frac{20+(\frac{16}{3}-5)\sqrt{15}-\frac{4}{3}(15)}{16-15}$$

$$= \frac{20+\frac{1}{3}\sqrt{15}-20}{16-15} = \frac{1}{3}\sqrt{15} \quad .$$

Ans. to 24.4: $x = \frac{1}{3}\sqrt{15}$.

How to convert A Quadratic Irrationality into an ultimately periodic continued fraction

Important Reminder:

$$\frac{1}{a+b\sqrt{Q}} = \frac{a-b\sqrt{Q}}{a^2-b^2Q} \quad .$$

So the reciprocal of an expression of the form $a+b\sqrt{Q}$ with a and b rational numbers and Q a positive integer, also has this form.

Problem 24.5: Convert $\sqrt{2}$ into an ultimately periodic continued fraction.

Sol. to 24.5: $\lfloor\sqrt{2}\rfloor = 1$, so we write

$$\sqrt{2} = 1 + (\sqrt{2}-1) \quad .$$

Now

$$\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}^2-1} = 1+\sqrt{2}$$

and we have, so far

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} \quad .$$

$\lfloor 1+\sqrt{2}\rfloor = 2$, so we write

$$1+\sqrt{2} = 2+(\sqrt{2}-1) == 2+\frac{1}{\frac{1}{\sqrt{2}-1}} = 2+\frac{1}{1+\sqrt{2}} \quad .$$

But this is *self-similar*. Writing $x = 1 + \sqrt{2}$, we have the relation

$$x = 2 + \frac{1}{x} \quad .$$

In other words $x = [2^\infty]$. Going back to $\sqrt{2} = 1 + 1/x$, we have:

Ans. to 24.5: $\sqrt{2} = [1, 2^\infty]$.

A more efficient way to find the continued fraction.

Go to maple, and do

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convert(x,confrac);
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detect a period, and prove it using the previous type of problem (converting an ultimately periodic infinite continued fraction into a quadratic irrationality).