Dr. Z.'s Number Theory Lecture 23 Handout: Integer Partitions III

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How to compute p(n) Fast

The generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} \quad ,$$

is **very** inefficient for computing the first 100 (or, on a computer, the first 1000000) values of $p(n)$. Thanks to Leonhard Euler, there is a much better way

Euler's Recurrence for p(n) If $n < 0$, then $p(n) = 0$. If $n = 0$, $p(0) = 1$, if $n > 0$, then

$$p(n) = -\sum_{j=1}^{\infty}(-1)^j p(n - j(3j-1)/2) - \sum_{j=1}^{\infty}(-1)^j p(n - j(3j+1)/2)$$

Note: The upper limit of summation above are ∞, but it is only for convenience. The actual summation is for j for which $j(3j+1)/2 \le n$, since after that the values are all 0.

Problem 23.1: Use Euler's recurrence to compile a table of $p(n)$ for $0 \le n \le 7$

Sol. to 23.1: It convenient first to put a list of the **pentagonal numbers**:

$$1, 5, 12, 22 \ldots \quad ; \quad 2, 7, 15, 26 \ldots$$

By definition:

$$p(0) = 1 \quad .$$

Now

$$p(1) = -(-1)^1 p(1 - 1 \cdot 2/2) = 1 \quad ,$$

$$p(2) = -(-1)^1 p(2 - 1 \cdot 2/2) - (-1)^1 p(2 - 1 \cdot 4/2) = p(1) + p(0) = 2 \quad .$$

$$p(3) = -(-1)^1 p(3 - 1 \cdot 2/2) - (-1)^1 p(3 - 1 \cdot 4/2) = p(2) + p(1) = 3 \quad .$$

$$p(4) = -(-1)^1 p(4 - 1 \cdot 2/2) - (-1)^1 p(4 - 1 \cdot 4/2) = p(3) + p(2) = 5 \quad .$$

$$p(5) = -(-1)^1 p(5-1\cdot2/2) - (-1)^2 p(5-2\cdot5/2) - (-1)^1 p(5-1\cdot4/2) = p(4) - p(0) + p(3) = 5-1+3 = 7 \quad .$$

$$p(6) = -(-1)^1 p(6-1\cdot2/2) - (-1)^2 p(6-2\cdot5/2) - (-1)^1 p(6-1\cdot4/2) = p(5) - p(1) + p(4) = 7-1+5 = 11 \quad .$$

$$p(7) = -(-1)^1 p(7 - 1 \cdot 2/2) - (-1)^2 p(7 - 2 \cdot 5/2) - (-1)^1 p(7 - 1 \cdot 4/2) - (-1)^2 p(7 - 2 \cdot 7/2)$$

$$= p(6) - p(2) + p(5) - p(0) = 11 - 2 + 7 - 1 = 15 \quad .$$

The Bressoud-Zeilberger Proof of Euler's Recurrence

An equivalent formula is as follows. Let $a(j) = (3j^2 + j)/2$. Then

$$\sum_{j\ even}^{\infty} p(n - a(j)) = \sum_{j\ odd}^{\infty} p(n - a(j)) \quad .$$

Read **crarefully** `http://www.math.rutgers.edu/~zeilberg/mamarimY/Zeilberger_y1985_p54.pdf`.

Problem 23.2: Apply the Bressoud-Zeilberger mapping ϕ (when $n = 61$), to:

$$j = 2 \quad , \quad \lambda = (14, 12, 12, 11, 5) \quad ,$$

and then apply it again and make sure that you get it back.

Solution to 23.2: Here, the **number of parts**, t, equals 5, and the **largest part**, $\lambda(1)$, equals 14. Since $t + 3j = 5 + 6 = 11$ is **less** than $\lambda(1) = 14$, we apply the second case and get that

$$\phi((14, 12, 12, 11, 5)) = (12 + 1, 12 + 1, 11 + 1, 5 + 1, 1, 1) = (13, 13, 12, 6, 1, 1) \quad .$$

And now $j = 2 + 1 = 3$.The number of 1's at the end being $\lambda(1) - (t + 3j) - 1 = 14 - 11 - 1 = 2$.

Now let's apply ϕ again. For the new partition $j = 3$, and $\lambda = (13, 13, 12, 6, 1, 1)$. **Now** $\lambda(1) = 13$ and $t = 6$. $6 + 3j = 6 + 3 \cdot 3 = 15$ is $\geq$ to $\lambda(1) = 13$ so the first case applies and since $t + 3j - 1 = 14$, we get

$$\phi((13, 13, 12, 6, 1, 1)) = (14, 13 - 1, 13 - 1, 12 - 1, 6 - 1, 1 - 1, 1 - 1) = (14, 12, 12, 11, 5) \quad .$$

(Of course we discard the 0-s). **Yea** we got it back.

Euler's Pentagonal Theorem

$$\prod_{i=1}^{\infty} (1 - q^i) = \sum_{j=-\infty}^{\infty} (-1)^j q^{(3j-1)j/2} \quad .$$

Fabian Frankin's beautiful proof

Read **carefully** `http://en.wikipedia.org/wiki/Pentagonal_number_theorem`.

Problem 23.3: Apply Franklin's mapping, F, if possible, to the distinct partition

$$\lambda = (8, 7, 6, 4, 3, 2) \quad .$$

Then apply it again and make sure that you get it back.

Sol. to 23.3:

The Ferrers graph is

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The length of the downward-diagonal from the top-right is 3, while the smallest part is 2. So it legal to take the bottom row (the smallest part) and make it into a new downward-diagonal from the top-right. e Ferrers graph is

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So

$$F((8,7,6,4,3,2)) = (9,8,6,4,3) \quad .$$

For $F(9,8,6,4,3)$, the size of the downward-diagonal from the top-right is 2, while the smallest part is 3. So we remove 1 from the top two parts, and stick 2 dots at the bottom, getting

$$F(9,8,6,4,3) = (8,7,6,4,3,2) \quad .$$