Dr. Z.'s Number Theory Lecture 1 Handout

By Doron Zeilberger

Natural Numbers

The most **natural** way to represent *natural numbers* is in *unary* notation

$$1 = 1 \quad , \quad 2 = 11 \quad , \quad 3 = 111 \quad , \quad 4 = 1111 \quad , \quad \ldots$$

How to add two natural numbers?

To find $a + b$, simply write-down a (in unary!) followed by b (in unary!).

How to multiply two natural numbers?

The **definition** of the product $P(a, b)$ is:

$$P(a, 0) = 0 \quad , \quad P(a, b + 1) = P(a, b) + a \quad .$$

To find $a \times b$, write a and b, given in unary, and then copy-and-paste a, each time deleting one dot from b, until there are no more dots in the b section.

Problem 1.1: Using *unary* (no credit for other methods!) Find

111 + 11111

Ans. to Problem 1.1: 11111111 .

Problem 1.2: Using *unary* (no credit for other methods!) Find

111 × 1111

Sol. to Problem 1.2:

Step 1: Write down

$$a = 111$$

$$b' = 1111$$

Step 2a: Since b' is not yet empty, write down a, and remove one 1 from b'

$$Ans = 111$$

$$b' = 111$$

Step 2b: Since b' is not yet empty, add a to Ans and remove one dot from b'

$$Ans = 111111$$

$$b' = 11$$

Step 2c: Since b' is not yet empty, add a to Ans and remove one dot from b'

$$Ans = 111111111$$

$$b' = 1$$

Step 2d: Since b' is not yet empty, add a to Ans and remove one dot from b'

$$Ans = 111111111111$$

$$b' =$$

Step 2e: Since b' is now empty, Ans' is the **final answer**.

Ans. to 1.2: 111111111111

von Neumann's representation of Natural Numbers in terms of sets

Some people believe that **sets** are more fundamental than (even natural!) *numbers*.

Here is how John von Neumann defines them

$$S(0) := \{\}$$

(the **empty set**)

$$S(1) = \{\{\}\} \quad ,$$

$$S(2) := \{S(0), S(1)\} = \{\{\}, \{\{\}\}\}$$

and in general, if $S(n)$ is the set representing n

$$S(n) := \{S(0), S(1), S(2), \dots, S(n-1)\}$$

Note: The number of elements of $S(n)$ is n

Problem 1.3: Write 4 in von-Neumann notation.

Ans. to 1.3:

$$0 := \{\}$$

$$1 := \{\{\}\} \quad ,$$

$$2 := \{\{\}, \{\{\}\}\} \quad ,$$

$$3 := \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$$

$$4 := \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\} \quad .$$

Ans. to Problem 1.3: The natural number 4 is written, in von-Neumann notation, is

$\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\}$.

Frege's (VERY ABSTRACT) definition of Natural Number

Two sets are *equivalent* if there is a **bijection** between them, i.e. one-to-one correspondence.

For example, if you have a club that only admits married couples, there is a bijection between the set of men and women called "wife of" and "husband of"

An *equivalence class* is the class of all sets that are equivalent to each other

Definition: A Natural number is an equivalence class of sets under the equivalence relation "being in bijection".

So the number 5 is, according to Frege, "the class of all sets with five elements".

For example, *three* **is** the class

$$\{\{a, b, c\}, TheThreeStoogers, TheTrinity, \{Yes, No, MayBe\}, \ldots\} \quad .$$

Problem 1.4: List some **natural** members of Frege's class representing two

Sol. to Problem 1.4: There are, of course, infinitely many answers. Here are two

$$\{Male\,,\, Female\}, \{Odd\,,\, Even\}, \{Ying\,,\, Yang\} \quad .$$

Peano's Axioms

1. Zero is a number.

2. If a is a number, the successor of a is a number.

3. zero is not the successor of a number.

4. Two numbers of which the successors are equal are themselves equal.

5. (induction axiom.) If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.