Solutions to MATH 477 (3), Dr. Z.'s , Final Exam, Monday, Dec. 18, 2017, 8:00-11:00am, PH 111

PUT The FINAL ANSWER TO EACH PROBLEM IN THE AVAILABLE BOX

1. (12 pts.) Suppose that it is known that the amount of gold dug in a Gold mine during one day is a random variable with mean 100 kg and variance 25 kg^2. what can be said about the probability of a day's production will be between 80 and 120 kg?

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ans. $\geq \frac{15}{16}$

Take $\mu = 100$, $\sigma = 5$, and $k = 20$ in **Chebychev's inequality**

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2} \quad ,$$

getting

$$P\{|X - 100| \geq 20\} \leq \frac{5^2}{20^2}$$

Hence

$$P\{|X - 100| \geq 20\} \leq \frac{1}{16} \quad .$$

Hence

$$P\{|X - 100| \leq 20\} \geq 1 - \frac{1}{16} = \frac{15}{16} \quad .$$

But $|X - 100| \leq 20$ is the same as $80 < X < 120$, so we are done.

2. (12 pts.) The moment generating function of a certain random variable, X, is $M_X(t) = e^{t^2/2+t^3/6}$. Calculate the third moment.

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ans. 1

$$M_X(t) = e^{t^2/2+t^3/6} = 1 + (t^2/2 + t^3/6) + (t^2/2 + t^3/6)^2/2 + \ldots = 1 + t^2/2 + t^3/6 + \ldots$$

(we don't care about the t^4 terms and beyond). By definition

$$M_X(t) = 1 + m_1\, t + \frac{m_2}{2!}\, t^2 + + \frac{m_3}{3!}\, t^3 + \dots \quad .$$

Comparing the coefficient of t^3, we get

$$\frac{m_3}{6} = \frac{1}{6} \quad ,$$

hence $m_3 = 1$.

3. (12 pts.) In the sample space $\{-1, 0, 1, 2\}$, with $Pr(-1) = Pr(0) = Pr(1) = Pr(2) = \frac{1}{4}$ define the random variables $X(i) = i, Y(i) = -i^2$. Find the correlation $\rho(X, Y)$.

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ans. $-\frac{\sqrt{5}}{3}$ or $-0.7453559923\dots$.

$$X(-1) = -1 \quad , \quad X(0) = 0 \quad , \quad X(1) = 1 \quad , \quad X(2) = 2 \quad ,$$

$$Y(-1) = -1 \quad , \quad Y(0) = 0 \quad , \quad Y(1) = -1 \quad , \quad Y(2) = -4 \quad ,$$

$$E[X] = (-1 + 0 + 1 + 2)/4 = \frac{1}{2} \quad , \quad E[X^2] = ((-1)^2 + 0^2 + 1^2 + 2^2)/4 = \frac{3}{3} \quad ,$$

$$E[Y] = (-1 + 0 - 1 - 4)/4 = -\frac{3}{2} \quad , \quad E[Y^2] = ((-1)^2 + 0^2 + (-1)^2 + (-4)^2)/4 = \frac{9}{2} \quad ,$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{3}{2} - (\frac{1}{2})^2 = \frac{5}{4} \quad , \quad Var(Y) = E[Y^2] - E[Y]^2 = \frac{9}{2} - (-\frac{3}{2})^2 = \frac{9}{4} \quad ,$$

$$E[XY] = ((-1)(-1) + (0)(0) + (1)(1) + (2)(2)/4 = -2 \quad ,$$

$$Cov(X, Y) = E[XY] \ - \ E[X]\, E[Y] = -2 - (\frac{1}{2})(-\frac{3}{2}) = -2 + \frac{3}{4} = -\frac{5}{4} \quad .$$

Finally

$$\rho(X, Y) \ = \ \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} \ = \ \frac{-\frac{5}{4}}{\sqrt{\frac{5}{4}}\sqrt{\frac{9}{4}}} = -\frac{\sqrt{5}}{3} \quad .$$

4. (12 pts.) The return on three investments, X and Y, Z, follows the joint probability density function

$$f(x,y,z) = \begin{cases} 8\,xyz \quad , & if \quad 0 \le x \le 1 \quad and \quad 0 \le y \le 1 \quad and \quad 0 \le z \le 1; \\ 0 \quad , & otherwise. \end{cases}$$

Calculate $E[6\,X^2Y^2Z]$.

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ans. 1 .

$$E]6X^2Y^2Z] = \int_0^1 \int_0^1 \int_0^1 (8xyz) \cdot (6x^2y^2z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^1 48x^3y^3z^2\,dx\,dy\,dz$$

$$= 48 \left(\int_0^1 x^3\,dx \right) \left(\int_0^1 y^3\,dy \right) \left(\int_0^1 z^2\,dz \right) = 48 \left(\frac{x^4}{4} \Big|_0^1 \right) \left(\frac{y^4}{4} \Big|_0^1 \right) \left(\frac{z^3}{3} \Big|_0^1 \right)$$

$$= 48 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = 1 \quad .$$

5. (12 pts.) A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy.

Let X denote the proportion of employees who purchase the basic policy, and let Y the proportion of the employees who purchase the supplemental policy. Let X and Y have joint density function $f(x,y) = 2(x+y)$ on the region where the density is positive.

Given that 30% of the employees buy the basic policy, determine the probability that fewer than 20% buy the supplemental policy.

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ans. $\frac{16}{27}$ or $0.5925925926\ldots$.

$$\frac{\int_0^{0.2} 2(0.3+y)\,dy}{\int_0^{0.3} 2(0.3+y)\,dy} .$$

The **top** is

$$\int_0^{0.2} 2(0.3+y) \;=\; \left(0.6\,y+y^2\right)\Big|_0^{0.2} = (0.6)\cdot(0.2)+(0.2)^2 \;=\; 0.16 \quad .$$

The **bottom** is

$$\int_0^{0.3} 2(0.3+y) \;=\; \left(0.6\,y+y^2\right)\Big|_0^{0.3} = (0.6)\cdot(0.3)+(0.3)^2 \;=\; 0.27 \quad .$$

Hence the desired answer is $\frac{0.16}{0.27} = \frac{16}{27}$.

6. (8 pts.) State the axioms of probability for finite sample spaces.

Given a finite set Ω called the *sample space*, P is a function on the set of subsets (denoted by 2^{Ω}), satisfying

- $P(\Omega) = 1$

- $0 \leq P(A) \leq 1$, for every subset A of Ω.

- If A and B are **disjoint** (i.e. $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B) \quad .$

7. (12 pts.) Let X be a random variable with probability density function

$$f(x) \;=\; \begin{cases} 3x^2 & , if \quad 0 \leq x \leq 1\,; \\ 0 & , otherwise. \end{cases}$$

Find the standard-deviation of X.

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ans. $\sqrt{\frac{3}{80}}$ or $0.1936491673\ldots$.

$$E[X] = \int_0^1 x\cdot(3x^2)\,dx = \int_0^1 3x^3\,dx = \left(3\frac{x^4}{4}\right)\Big|_0^1 = \frac{3}{4} \quad .$$

$$E[X^2] = \int_0^1 x^2\cdot(3x^2)\,dx = \int_0^1 3x^4\,dx = \left(3\frac{x^5}{5}\right)\Big|_0^1 = \frac{3}{5} \quad .$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = \frac{3}{80} \quad .$$

Finally

$$SD(X) = \sqrt{Var(X)} = \sqrt{\frac{3}{80}} \quad .$$

8. (12 pts.) The monthly profit of Company I can be modeled by a continuous random variable with density function $f(x)$. Company II has a monthly profit that is the cube of that of Company I.

Determine the probability density function, call it $g(y)$, of the monthly profit of Company II.

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ans. $\frac{1}{3}y^{-\frac{2}{3}}f(y^{\frac{1}{3}})$.

Let $G(y)$ be the cdf of Y. By definition

$$G(y) = P\{Y \leq y\} \quad .$$

But $Y = X^3$, hence

$$G(y) = P\{X^3 \leq y\} = P\{X \leq y^{\frac{1}{3}}\} = F(y^{\frac{1}{3}}) \quad ,$$

(by definition of the cdf of X). Hence

$$G(y) = F(y^{\frac{1}{3}}) \quad .$$

Take derivative w.r.t. to y, we get (since the pdf, $g(y)$ is the derivative of the cdf $G(y)$)

$$g(y) = G'(y) = \left(F(y^{\frac{1}{3}})\right)' \quad .$$

By the **chain rule** (in its semi-abstract setting) this equals

$$F'(y^{\frac{1}{3}})(y^{\frac{1}{3}})' = f(y^{\frac{1}{3}}) \cdot \frac{1}{3}y^{-\frac{2}{3}} \quad .$$

9. (12 pts.) The probability that a randomly chosen dog is a biter is 0.2. Dogs who are biters are three times as likely to be barkers as those who do not bite.

What is the conditional probability that a randomly chosen dog is a biter, given that it is a barker?

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ans. $\frac{3}{7}$.

We have $P(Biter) = 0.2$ and $P(NonBiter) = 0.8$.

Let $P(Barker|NonBiter) = p$. Then $P(Barker|Biter) = 3p$.

By **Bayes**

$$P(Biter|Barker) = \frac{P(Barker|Biter) \cdot P(Biter)}{P(Barker|Biter) \cdot P(Biter) + P(Barker|NonBiter) \cdot P(NonBiter)}$$

$$\frac{(0.2) \cdot (3p)}{(0.2) \cdot (3p) + (0.8) \cdot (p)} = \frac{(0.6)p}{(0.6 + 0.8)p} = \frac{6}{14} = \frac{3}{7} \quad .$$

10. (12 pts.) What is the probability that among 3 people, at least two of them have the same birth-month? (Assuming (a little stupidly, since Aug. is more likely than Feb.) that all 12 months are equally likely).

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ans. $\frac{17}{72}$ or $0.2361111111\ldots$.

$$1 - 1 \cdot (1 - \frac{1}{12}) \cdot (1 - \frac{2}{12}) = 1 - \frac{11 \cdot 10}{12^2} = \frac{17}{72} \quad .$$

11. (12 pts.) It is known that the probability of being lazy equals twice the probability of being smart, and the probability of being smart is twice the probability of being strong. All three traits are independent of each other. If the probability of being

lazy **and** smart **and** strong is $\frac{8}{1000}$, what is the probability of being lazy?

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ans. $\frac{2}{5}$ or 0.4 .

Let $P(Strong) = x$. Then $P(Smart) = 2x$ and $P(Lazy) = 2P(Smart) = 2(2x) = 4x$.

By **independence**

$$P(Strong \quad AND \quad Smart \quad AND \quad Lazy) - x(2x)(4x) = 8x^3 \quad .$$

According to the question, this equals $\frac{8}{1000}$, hence

$$8x^3 = \frac{8}{1000} \quad ,$$

giving $x = \frac{1}{10}$, hence $P(Lazy) = 4 \cdot \frac{1}{10} = \frac{2}{5}$.

12. (12 pts.) Suppose that $Var(X) = 1, Var(Y) = 3, Var(Z) = 5$, $Var(X+Y) = 6, Var(X+Z) = 10, Var(Y+Z) = 10$. Find $Var(X - Y + Z)$.

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ans. 9 .

$$Cov(X,Y) = \frac{Var(X+Y) - Var(X) - Var(Y)}{2} = \frac{6-1-3}{2} = 1 \quad .$$

$$Cov(X,Z) = \frac{Var(X+Z) - Var(X) - Var(Z)}{2} = \frac{10-1-5}{2} = 2 \quad .$$

$$Cov(Y,Z) = \frac{Var(Y+Z) - Var(Y) - Var(Z)}{2} = \frac{10-3-5}{2} = 1 \quad .$$

Next

$$Var(X-Y+Z) = Var(X)+Var(-Y)+Var(Z)+2Cov(X,-Y)+2Cov(X,Z)+2Cov(-Y,Z)$$

$$Var(X) + (-1)^2 \cdot Var(Y) + Var(Z) - 2Cov(X,Y) + 2Cov(X,Z) - 2Cov(Y,Z)$$

$$= Var(X) + Var(Y) + Var(Z) - 2Cov(X,Y) + 2Cov(X,Z) - 2Cov(Y,Z) =$$

$$1 + 3 + 5 - 2 \cdot 1 + 2 \cdot 2 - 2 \cdot 1 = 9 \quad .$$

13. (12 pts.) Let $X_1, \ldots, X_{300}$ be independent random variables, each with (the same) density function given by:

$$f(x) = \begin{cases} 2x & , \quad if \quad 0 \leq x \leq 1\,; \\ 0 & , \quad otherwise \end{cases} \quad ,$$

and let $X = \sum_{i=1}^{300} X_i$.

Use the Central Limit Theorem to estimate the probability that $X < 202$. You can leave your answer in terms of the Φ function (the cdf of the standard normal distribution)

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ans. $\Phi(\frac{\sqrt{6}}{5})$ or $0.6878969426\ldots$.

$$E[X_i] = \int_0^1 2x^2\,dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3} \quad .$$

$$E[X_i^2] = \int_0^1 2x^3\,dx = 2\frac{x^4}{4}\Big|_0^1 = \frac{1}{2} \quad .$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18} \quad .$$

By **Linearity of Expecation**

$$E[X] = 300 \cdot \frac{2}{3} = 200 \quad .$$

By **Linearity of Variance** (for independent random variables)

$$Var(X) = 300 \cdot \frac{1}{18} = \frac{50}{3} \quad .$$

Hence $\sigma = \sqrt{\frac{50}{3}}$.

By the **Central Limit Theorem** X is approximated by $N(200, (\sqrt{\frac{50}{3}})^2)$.

$$P\{X < 202\} = P\{X-200 < 202-200\} = P\{X-200 < 2\} = P\{\frac{X-200}{\sigma} < \frac{2}{\sigma}\} = \Phi(\frac{2}{\sqrt{(50/3)}}$$

$$= \Phi(\frac{\sqrt{6}}{5}) \quad .$$

14. (12 pts.) (a) (6 points) State Chebychev's Inequality ; (b) (6 points) State the weak Law of Large Numbers.

(a) If X is random variable with mean μ and variance σ^2, then for any value $k > 0$,

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2} \quad .$$

(b) Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables, each having a finite mean $E[X_i] = \mu$. Then for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\{\,|\frac{X_1 + \ldots + X_n}{n} - \mu| \geq \epsilon\,\} = 0 \quad .$$

15. (12 pts.) You toss a fair coin three times.

If you get 3 Heads you win 2000 dollars.If you get 2 Heads you win 1000 dollars.If you get 1 Heads you lose 600 dollars. If you get 0 Heads you lose 1400 dollars.

What is your expected gain, if it is known that you did **not** get three Heads.

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ans. $-\frac{200}{7}$.

$$\frac{\frac{3}{8}\cdot(1000) + \frac{3}{8}\cdot(-600) + \frac{1}{8}\cdot(-1400)}{\frac{3}{8}+\frac{3}{8}+\frac{1}{8}} = \frac{3\cdot 1000 - 3\cdot 600 - 1\cdot 1400}{7} = -\frac{200}{7} \quad .$$

16. (12 points) State (3 points) and prove (9 points) Markov's inequality.

Statement

If X is a random variable that takes only non-negative values, then for any $a > 0$:

$$P\{X \geq a\} \leq \frac{E[X]}{a} \quad .$$

Proof

For $a > 0$, let

$$I = \begin{cases} 1 \;, & if \quad X \geq a\,; \\ 0 \;, & otherwise. \end{cases} \quad .$$

Obviously (since $X \geq 0$)

$$I \leq \frac{X}{a} \quad .$$

(When $X < a$ the left side is 0 and the right side is non-negative, when $X \geq a$ the left side is 1 and the right side is ≥ 1 (and usually, > 1).

Taking expectation, we have

$$E[I] \leq E[\frac{X}{a}] = \frac{E[X]}{a} \quad .$$

But $E[I] = 0 \cdot P[X < a] + 1 \cdot P[X \geq a] = P[X \geq a]$, so the inequality follows.

17. (12 pts.) You are visiting the Royal Gardens in London, England, that is a huge maze. There are no people to ask directions, and no maps (or gps).

The garden has two statues, Queen Victoria and Queen Mary. You are currently next to Queen Victoria, and you are desperate to get out of the maze.

• If you are at the Queen Victoria statue, there are two paths. One path leads out of the maze, and takes 9 minutes to walk, the second takes 3 minutes to walk and takes you to the statute of Queen Mary. You are equally likely to take either path (if you are currently at Queen Victoria).

• If you are at the Queen Mary statue, there are two paths. One path leads out of the maze, and takes 12 minutes, the second takes 6 minutes and takes you to the statute of Queen Victoria. You are equally likely to take either path (if you are currently at Queen Mary).

What is the expected time of getting out of the maze, if you are currently next to the statute of Queen Victoria?

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ans. 14 minutes.

Let

• V be the expected time for getting our of the maze if you are currently next to Queen Victoria

• M be the expected time for getting our of the maze if you are currently next to Queen Mary

If you are at Queen Victoria

your probability is $\frac{1}{2}$ of getting immediately out (contributing $\frac{1}{2} \cdot 9$ to the expected time it takes to get out)

your probability is $\frac{1}{2}$ of going to Queen Mary (contributing $\frac{1}{2} \cdot (3 + M)$ to the expected time it takes to get out).

Hence, we have

$$V = \frac{9}{2} + \frac{1}{2}(M + 3) \quad .$$

If you are at Queen Mary,

your probability is $\frac{1}{2}$ of getting immediately out (contributing $\frac{1}{2} \cdot 12$ to the expected time it takes to get out)

your probability is $\frac{1}{2}$ of going to Queen Victoria (contributing $\frac{1}{2} \cdot (6 + V)$ to the expected time it takes to get out). Hence, we have

$$M = \frac{12}{2} + \frac{1}{2}(V + 6) \quad .$$

We get two equations for the two unknowns, V and M.

$$V - \frac{1}{2}M = 6 \quad , \quad M - \frac{1}{2}V = 9 \quad .$$

From the second equation, we get $M = 9 + \frac{1}{2}V$. Putting it in the first equation, we get

$$V - \frac{1}{2}(9 + \frac{V}{2}) = 6 \quad ,$$

hence

$$\frac{3}{4}V = 6 + \frac{9}{2} = \frac{21}{2}$$

giving $V = \frac{4}{3} \cdot \frac{21}{2} = 14$.