Dr. Z.'s Probability Lecture 9 Handout: The Poisson Random Variable

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Very Important Discrete Random Variable: A random variable X that takes on the values $0, 1, 2, \ldots$ is a **Poisson Random Variable** with **parameter** λ , for some $\lambda > 0$, if its probability mass function is given by

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad .$$

Notes: **1.** Since we know from calculus that $e^\lambda = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$, we have that $\sum_{i=0}^{\infty} P(X = i) = 1$, so all the probabilities add up to 1, as they should.

2. The Poisson distribution with parameter λ is the limit of the Binomial distribution with parameters $(n, \frac{\lambda}{n})$ as n gets very large. It is so common since it models "accidents". If a city, on average, has two car accidents per day, and there are a million cars, the probability that any one car will have an accident is very small, and they are all independent, but since there are so many cars, you would expect 2 accidents, on average. But sometimes there are no accidents, and sometimes there are 3, and 4, and rarely, 5. Also the number of spam emails and many many other things are Poisson! (In particular, the number of Prussian officers killed by their horses.)

Important Facts: If X is a Poisson random variable with parameter λ, then

$$E[X] = \lambda \quad , \quad Var(X) = \lambda.$$

Note: They happen to be the same! This hardly ever happens. It is a lucky coincidence. So this is not a typo!

Warning: Since the variance is λ, the standard deviation is $\sqrt{\lambda}$. Do not confuse variance and standard deviation.

Another important fact: If X is a Poisson random variable with parameter λ, then the second moment, $E[X^2]$, is given by:

$$E[X^2] = \lambda(\lambda + 1) \quad .$$

Note: This follows from the fact that $Var(X) = E[X^2] - E[X]^2$.

Important Fact: If X_1 and X_2 are Poisson random variables with parameters λ_1 and λ_2, and they are **independent** of each other, then $X_1 + X_2$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

More generally: If $X_1, X_2, \ldots, X_m$ are Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ and each of them is independent of all the other, then their sum, $X_1 + \ldots + X_m$ is yet another Poisson random variable with parameter $\lambda_1 + \ldots + \lambda_m$.

Problem 9.1: The expected number of phone calls I get in a day is a Poisson random variable with average 3.2.

What is the probability that tomorrow I will get

(i) 0 phone calls (ii) at least 2 phone calls?

Sol. to 9.1(i): Since $\lambda = 3.2$,

$$P(X = 0) = (3.2)^0 \cdot e^{-3.2}/0! = 0.04076220398 \quad .$$

Sol. to 9.1(ii): Since $\lambda = 3.2$,

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - ((3.2)^0 \cdot e^{-3.2}/0! + (3.2)^1 \cdot e^{-3.2}/1!) = 1 - exp(-3.2) \cdot (1 + 3.2) = 0.8287987433\ldots \quad .$$

Ans. to 9.1: The probability that I will get no phone calls is %4.076220..., and the probability that I will get at least two phone calls is %82.879874....

Problem 9.2: The number of dinner customers in a certain restaurant on Tuesday is a Poisson random variable with mean 10. If you are told that this Tuesday they had strictly more than 10 customers, what is the probability that they had strictly less than 15 customers?

Sol. to 9.2:

$$P(X \leq 14 \,|\, X \geq 11) = \frac{P(11 \leq X \leq 14)}{P(X \geq 11)} \quad .$$

The denominator is

$$P(X \geq 11) = 1 - P(X \leq 10) = 1 - e^{-10} \left(\sum_{i=0}^{10} \frac{10^i}{i!} \right) = 0.416960250\ldots \quad .$$

The numerator is

$$e^{-10} \left(\sum_{i=11}^{14} \frac{10^i}{i!} \right) = e^{-10} \left(\frac{10^{11}}{11!} + \frac{10^{12}}{12!} + \frac{10^{13}}{13!} + \frac{10^{14}}{14!} \right) = 0.333501\ldots \quad .$$

Hence the desired conditional probability is

$$\frac{0.333501}{0.416960250} = 0.7998406968\ldots \quad .$$

Ans. to 9.2: The conditional probability of the restaurant having strictly less than 15 customers, granted that they had strictly more than 10 customers is %79.98406968... .

Problem 9.3 Compare the Poisson approximation with the correct binomial distribution for the following cases

(i) $P[X = 2]$ when $n = 100$ and $p = .01$;

(ii) $P[X = 1]$ when $n = 100$ and $p = .03$.

Sol. to 9.3(i): The exact probability is

$$\binom{100}{2}(0.01)^2(0.99)^{98} = 0.1848648188\ldots \quad .$$

Since the expectation of the Binomial random variable is $100 \cdot 0.01 = 1$, we approximate it with a Poisson random variable with parameter 1. So

$$P(X = 2) \quad is \quad approximately \quad e^{-1}\frac{1^2}{2!} = 0.1839397206\ldots .$$

Not bad!

Sol. to 9.3(ii): The exact probability is

$$\binom{100}{1}(0.03)^1(0.97)^{99} = 0.1470696122\ldots \quad .$$

Since the expectation of the Binomial random variable is $100 \cdot 0.03 = 3$, we approximate it with a Poisson random variable with parameter 3. So

$$P(X = 1) \quad is \quad approximately \quad e^{-3}\frac{3^1}{1!} = 0.1493612051\ldots .$$

Still not bad, but not as good.

Problem 9.4: The number of students who enroll in my advanced graduate course each Spring semester is a Poisson distribution with mean 6. The enrollment each Spring is independent of the enrollment of those of other Springs.

What is the probability that I will get at least 19 students altogether in the next three Spring semesters?

Sol. to 9.4: Since the sum of Poisson random variables that are *independent* is still a Poisson random variable with the parameters adding up, the total enrollment, in the next three Spring semesters is a Poisson random variable with parameter 18. Hence the desired probability is

$$P(X > 18) = 1 - P(X \leq 18) = 1 - e^{-18}\left(\sum_{i=0}^{18}\frac{18^i}{i!}\right) = 0.4377550141 \quad .$$

Ans. to 9.4: The probability that Dr. Z. will get at least 19 students altogether in the next three Spring semesters is %43.77550141 . . ..

Problem 9.5: A traffic expert in a certain large city discovers that is it five times more likely to have three accidents than to have five accidents. If the number of accidents in that city is a Poisson distribution, what is the variance of the number of accidents?

Sol. to 9.5: The problem tells you that

$$\frac{P(X=3)}{P(X=5)} = 5 \quad .$$

If the parameter of the Poisson distribution is λ, then since $P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$, we have

$$\frac{e^{-\lambda}\frac{\lambda^3}{3!}}{e^{-\lambda}\frac{\lambda^5}{5!}} = \frac{20}{\lambda^2} \quad .$$

(Note that $e^{-\lambda}$ canceled out.) Hence

$$\frac{20}{\lambda^2} = 5$$

Doing the simple algebra we get $\lambda = 2$ (note that it must be positive!). Hence the expectation is 2. But no one asked us for the expectation, they asked for the variance. But for a Poisson random variable the variance is the same as the expectation, so the answer is that the variance is 2.

Ans. to 9.5: The variance of the number of accidents is 2.

Problem 9.6: Let X represent the number of Email messages that you get between midnight and noon and let Y represent the number of Emails you get after noon but before midnight.

You are given

- X and Y are Poisson distributed.
- The first moment of Y is one more than the first moment of X.
- The second moment of X is $\frac{2}{3}$ of the second moment of Y.

Calculate the variance of X.

Sol. to 9.6: Let x be the parameter (alias expectation, alias first moment) of X, and Let y be the parameter (alias expectation, alias first moment) of Y. The second moments are $x(x+1)$ and $y(y+1)$ respectively.

We know that

$$y = x+1 \quad , \quad x(x+1) = \frac{2}{3}y(y+1) \quad .$$

Since $y = x+1$, putting it in the second equation, we get

$$x(x+1) = \frac{2}{3}(x+1)(x+2) \quad .$$

So

$$3x(x+1) - 2(x+1)(x+2) = 0 \quad .$$

Factoring out $x + 1$, we get

$$(x + 1)(3x - 2x - 4) = 0 \quad ,$$

giving

$$(x + 1)(x - 4) = 0 \quad .$$

Since x must be positive, we get that $x = 4$. Hence the expectation of X is 4, and so is the variance [they are always (for Poisson random variables) the same]. So the answer is that the variance of X is 4.

Ans. to 9.6: The variance of X is 4.