Dr. Z.'s Probability Lecture 8 Handout: The Bernoulli and Binomial Random Variables

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Important Definition: The **Bernoulli** random variable with parameter p can only take **two** values, $X = 0$ (failure) and $X = 1$ (success). Its probability mass function is

$$p(0) = P(X = 0) = 1 - p \quad , \quad p(1) = P(X = 1) = p \quad .$$

Note: When you toss a coin whose probability is p of landing on Heads **one** time, and count the number of heads, it is a Bernoulli random variable.

Important Definition: The **Binomial** random variable with **parameters** (n, p) can take values, from $X = 0$ all the way to $X = n$, and its probability mass function is

$$P(X = i) = \binom{n}{i} p^i (1 - p)^{n-i} \quad , \quad i = 0, 1, \ldots, n \quad .$$

Note: When you toss a coin, whose probability is p of landing on Heads, n times, and count the number of heads, it is given by a Binomial random variable with parameters (n, p).

Note: a Binomial random variable with parameters $(1, p)$ is a Bernoulli random variable with parameter p.

Problem 8.1 In a multiple-choice test with 8 possible answers for each of the 20 questions, what is the probability that the student will get 5 or more correct answers just by guessing?

Sol. to 8.1: Whether you guess right any of the questions is **independent** of your successes in the other. The probability of success in one try is $p = \frac{1}{8}$. So this is a Binomial random variable with **parameters** $(20, \frac{1}{8})$ whose probability mass function is

$$P(X = i) = \binom{20}{i} (\frac{1}{8})^i (\frac{7}{8})^{20-i} \quad , \quad i = 0, 1, \ldots, n \quad .$$

We need

$$\sum_{i=5}^{20} P(X = i) \; = \; 1 - \sum_{i=0}^{4} P(X = i) \; = \; 1 - \sum_{i=0}^{4} \binom{20}{i} (\frac{1}{8})^i (\frac{7}{8})^{20-i} =$$

$$1 - \left(\binom{20}{0} (\frac{1}{8})^0 (\frac{7}{8})^{20} + \binom{20}{1} (\frac{1}{8})^1 (\frac{7}{8})^{19} + \binom{20}{2} (\frac{1}{8})^2 (\frac{7}{8})^{18} + \binom{20}{3} (\frac{1}{8})^3 (\frac{7}{8})^{17} + \binom{20}{4} (\frac{1}{8})^4 (\frac{7}{8})^{16} \right)$$

$$= 0.095011166 \ldots \quad .$$

Ans. to 8.1: The probability that the student will get 5 or more correct answers just by guessing is %9.5011166 . . . .

Problem 8.2 What is the probability that when you toss a fair die 20 times, the number of times it lends on either 5 or 6 is more than ten but less than fifteen?

Sol. to 8.2: Here "success" means landing on 5 or 6, so $p = \frac{1}{3}$. This is a binomial random variable with parameters $(20, \frac{1}{3})$. We need

$$\sum_{i=11}^{14} P(X = i) \;=\; \sum_{i=11}^{14} \binom{20}{i} (\frac{1}{3})^i (\frac{2}{3})^{20-i} =$$

$$\binom{20}{11} (\frac{1}{3})^{11} (\frac{2}{3})^9 + \binom{20}{12} (\frac{1}{3})^{12} (\frac{2}{3})^8 + \binom{20}{13} (\frac{1}{3})^{13} (\frac{2}{3})^7 + \binom{20}{14} (\frac{1}{3})^{14} (\frac{2}{3})^6$$
$$= 0.03746920514\ldots \quad .$$

Ans. to 8.2: The requested probability is $\%3.746920514\ldots$.

Problem 8.3: Every year Georgia plays against Georgia Tech. The probability of Georgia winning is 0.6. Assuming that the outcomes of the matches are independent from all other matches, what is the probability that Georgia Tech will win at least three games in a ten year period?

Sol. to 8.3: This is $B(10, 0.4)$ (the probability of Georgia Tech winning any one game is 0.4). We need

$$\sum_{i=3}^{10} P(X = i) = 1 - \sum_{i=0}^{2} P(X = i) = 1 - \sum_{i=0}^{2} \binom{10}{i} (0.4)^i (0.6)^{10-i}$$

$$= 1 - \left(\binom{10}{0} (0.4)^0 (0.6)^{10} + \binom{10}{1} (0.4)^1 (0.6)^9 + \binom{10}{2} (0.4)^2 (0.6)^8 \right) = 0.8327102464 \quad .$$

Ans. to 8.3: The probability of Georgia Tech winning at least 3 games is %83.271.

Problem 8.4: There are four parallel sections of Probability, each with 20 students. The probability of a student failing this class is 0.3. What is the probability that at least two sections had strictly more than 15 passing students?

Sol. of 8.4: This is a *multi-step* problem, where a Binomial random variable shows up **twice**.

A section is considered a success if it has at least 16 students passing. Each section is $B(20, 0.7)$ so

$$P(Section \quad Succeeds) = \sum_{i=16}^{20} \binom{20}{i} (0.7)^i (0.3)^{20-i} \;=\; 0.2375077789 \quad .$$

Now in the "meta contest", between sections, this is $B(4, 0.2375077789)$, and we need

$$\sum_{i=2}^{4} P(X = i) = 1 - \sum_{i=0}^{1} P(X = i) =$$

$$1-\left(\binom{4}{0}(0.2375077789)^0(1-0.2375077789)^4+\binom{4}{1}(0.2375077789)^1(1-0.2375077789)^3\right)$$
$$=0.2408235097 \quad .$$

Sol. to 8.4: The probability that at least two sections had strictly more than 15 passing students is $\%24.08235$.

Important Fact: If X is a Binomial random variable with parameters (n, p) then $E[X] = np$ and $Var(X) = np(1-p)$.

Problem 8.5: If you toss a coin whose probability of Heads is 0.3, 100 times, what is the expected number of Heads? What is the standard deviation?

Sol. to 8.5: $E[X] = 100 \cdot 0.3 = 30$, $Var(X) = 100 \cdot 0.3 \cdot 0.7 = 21$, hence $SD(X) = \sqrt{21} = 4.582575695$

Problem 8.6: Bob has a good day with probability 0.6. If a day is not good, then it is bad. At any given day, Bob has a good day independently of what happened in other days.

Last week, Bob had strictly more than three good days. How likely is it that he had at least one bad day?

Sol. to 8.6: This is a binomial random variable with parameters $(7, 0.6)$. Let X be the number of good days. We need (since if he had at least one bad day, he had at most six good days)

$$P(X \leq 6 \,|\, X > 3) = P(0 \leq X \leq 6 \,|\, 4 \leq X \leq 7) \quad .$$

By the formula for conditional probability

$$P(E|F) = \frac{P(EF)}{P(F)} \quad ,$$

we have

$$P(0 \leq X \leq 6 \,|\, 4 \leq X \leq 7) = \frac{P(4 \leq X \leq 6)}{P(4 \leq X \leq 7)}$$

Now, the numerator is

$$P(4 \leq X \leq 6) \;=\; P(X=4) + P(X=5) + P(X=6)$$
$$= \binom{7}{4}(0.6)^4(0.4)^3 + \binom{7}{5}(0.6)^5(0.4)^2 + \binom{7}{6}(0.6)^6(0.4)^1 \;=\; 0.6822144 \quad .$$

The denominator is

$$P(4 \leq X \leq 7) \;=\; P(4 \leq X \leq 6) + P(X=7)$$
$$= 0.6822144 + \binom{7}{7}(0.6)^7(0.4)^0 = 0.6822144 + 0.02799360 = 0.71020800$$

Hence, the desired conditional probability is

$$\frac{0.6822144}{0.71020800} = 0.9605839416 \quad .$$

Ans. to 8.6: The chance that Bob had at least one bad day last week, granted that he had strictly more than three good days is $\%96.058294\ldots$.