Dr. Z.'s Probability Lecture 5 Handout: Independent events

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Independent Events (two events) Two events E and F are **independent** if

$$P(EF) = P(E)P(F) \quad .$$

Note: Another way of saying the same thing is $P(E|F) = P(E)$, knowing that F happened does not change the chance that E happened.

Independent Events (three events) Three events E and F and G are **independent** if they are **pairwise** independent

$$P(EF) = P(E)P(F) \quad , \quad P(EG) = P(E)P(G) \quad , \quad P(FG) = P(F)P(G) \quad ,$$

and in addition

$$P(EFG) = P(E)P(F)P(G) \quad .$$

Warning: Just having $P(EFG) = P(E)P(F)P(G)$ is not enough!.

Problem 5.1: Suppose that you toss a loaded coin whose probability of Heads is .3, then roll a loaded die whose probability of landing on an even number of dots is 0.6. Suppose that your probability of getting a headache is 0.1. Assuming that all these events are independent, what is the probability that the coin landed on Tail, the die landed on an odd number and you did not get a headache?

Sol. to 5.1: The probability of the coin landing on Tail is $1 - 0.3 = 0.7$; the probability that the die landed on an odd number is 0.4; the probability that you did not get a headache is 0.9. Since the original events are independent of each other, so are their negations, so that desired probability is

$$0.7 \cdot 0.4 \cdot 0.9 \; = \; 0.252 \quad .$$

Problem 5.2: If you toss a loaded coin, whose probability of landing on Heads is p, 5 times, what is the probability of getting exactly 2 Heads.

Sol. to 5.2: Since the tosses are **independent** of each other, and $Pr(Tail) = 1-p$, the probability of the sequence $hhttt$, for example, is $p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) = p^2(1-p)^3$. Similarly, the probability of the sequence $httth$ is $p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot p = p^2(1-p)^3$. Since multiplication is **commutative**, the probability of **any** sequence of length 5 with 2 Heads and 3 Tails is $p^2(1-p)^3$. Since there are exactly $\binom{5}{2} = 10$ such sequences, the desired probability is $\binom{5}{2}p^2(1-p)^3$.

Generalizing, the same argument yields to:

Important Fact: If you toss a coin whose probability of Heads is p, n times, the probability that you got exactly k Heads is (the so-called **Binomial Distribution**)

$$\binom{n}{k} p^k (1-p)^{n-k} \quad .$$

Note: This applies to any situation where there is a **dichotomy**, success, failure, that are repeated n times, and one outcome is independent of all the rest.

Problem 5.3 If you roll a loaded die, whose faces are marked by $1, 2, 3, 4, 5, 6$, and such that

$$P(1) = p_1 \quad , \quad P(2) = p_2 \quad , \quad P(3) = p_3 \quad , \quad P(4) = p_4 \quad , \quad P(5) = p_5 \quad , \quad P(6) = p_6 \quad ,$$

(where, of course, $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$), n times, What is the probability that you get a_1-1's , a_2-2's , a_3-3's , a_4-4's ,a_5-5's , and a_6-6's (where of course $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = n$).

Sol. to 5.3: Since each toss is **independent** of the others, the probability of any one **specific** sequence with a_1-1's , a_2-2's , a_3-3's , a_4-4's ,a_5-5's , and a_6-6's is

$$p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} p_5^{a_5} p_6^{a_6} \quad .$$

But we know, from Lecture 1, that the number of sequences with these specs is the **multinomial coefficient**

$$\frac{n!}{a_1! a_2! a_3! a_4! a_5! a_6!} \quad ,$$

so the desired probability is

$$\frac{n!}{a_1! a_2! a_3! a_4! a_5! a_6!} p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} p_5^{a_5} p_6^{a_6} \quad .$$

Problem 5.4: It is known that the probability of being tall is 0.1 more than the probability of being nice, and the probability of being tall and nice is 0.2. Assuming that the traits of being tall and being nice are independent, what is the probability of being tall?

Sol. to 5.4: Let x be the probability of being nice. Then the probability of being tall is $x + 0.1$, and since these traits are **independent**, the probability of being tall and nice is $x(x + 0.1)$. The problem tells you that this equals 0.2. Hence we have to solve the **quadratic equation**

$$x(x + 0.1) = 0.2 \quad .$$

But $x^2 + 0.1x - 0.2 = (x - 0.4)(x + 0.5)$, so the roots are $x = 0.4$ and $x = -0.5$. The solution $x = -0.5$ is **nonsense** (since a probability must be ≥ 0 and ≤ 1), so $x = 0.4$. It follows that the probability of being nice is 0.4, and hence the probability of being tall is 0.5.

Ans. to 5.4: The probability of being tall is 0.5.

Note: The following problem belongs more to the Lecture 4, but it is a good problem.

Problem 5.5 A class has 200 students. All the students are either Juniors of Seniors, and either Female or Male.

Sixty percent of them are male. If a female is randomly selected from the class, the probability that she is a Junior is 0.3. There are 150 Seniors in the class. A Junior is randomly selected in the class. Calculate the probability that the person selected is female.

Sol. to 5.5. The first step is to figure out the following quantities

- "Number of Female Juniors", let's call it x_{FJ}
- "Number of Male Juniors", let's call it x_{MJ}
- "Number of Female Seniors", let's call it x_{FS}
- "Number of Male Seniors", let's call it x_{MS}

The number of Males is $200 \cdot 0.6 = 120$ hence the number of Females is $200 - 120 = 80$, so we know that

$$x_{MJ} + x_{MS} = 120 \quad , \quad x_{FJ} + x_{FS} = 80 \quad .$$

Also, since there are 150 Seniors (and hence 50 Juniors), we have

$$x_{MJ} + x_{FJ} = 50 \quad , \quad x_{MS} + x_{FS} = 150 \quad .$$

Since the total number of Females is 80, and the probability of a Female being a Junior is 0.3, it follows that the number of Female Juniors is $80 \cdot 0.3 = 24$. So we got that $x_{FJ} = 24$ and hence $x_{FS} = 80 - 24 = 56$. From $x_{MS} + x_{FS} = 150$, we get $x_{MS} = 150 - 56 = 94$, and finally either from $x_{MJ} + x_{FJ} = 50$, or from $x_{FJ} + x_{FS} = 80$, we get $x_{MJ} = 120 - 94 = 26$ (or $x_{MJ} = 50 - 24 = 26$).

So we got

$$X_{MJ} = 26 \quad , \quad x_{FJ} = 24 \quad ,$$

$$X_{MS} = 94 \quad , \quad x_{FS} = 56 \quad .$$

We can finally answer the question. The total number of Juniors is 50, and the number of Female Juniors is 24, hence the probability that a randomly chosen Junior is Female is $\frac{24}{50} = 0.48$.

Ans. to 5.5: The probability that a randomly chosen Junior is Female is %48.

Important Problem (Problem of the Points)

If Independent trials resulting in success with probability p and failure with probability $1 - p$ are performed, then **the probability that n successes occur before m failures** is

$$\sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k} \quad .$$

Explanation: If there are $m+n-1$ trials and k of them were successes the probability of that happening is $\binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$, if you add these up from $k=0$ to $k=m+n-1$, you get 1, of course (since the total probability is 1, and also because it is $(p+(1-p))^{m+n-1} = 1$, thanks to the Binomial theorem. The cases $k=n$, $k=n+1$, correspond to there being at least n successes, and of course, the number of failues had to be less than m.

Problem 5.6. Two soccer teams, A, and B compete. The probability that team A scores a goal is $\frac{2}{3}$. The team who scores any given goal is independent of who scored any other goal.

Calculate the probability that team B scored 4 goals before team A's 3^{rd} goal.

Sol. to 5.6: Here "success" means team B scoring, and "failure" means A scoring, so $p = \frac{1}{3}$, $n = 4$ and $m = 3$. Plugging into the formula, we have

$$\sum_{k=4}^{6} \binom{6}{k} (\frac{1}{3})^k (\frac{2}{3})^{6-k} = \binom{6}{4} (\frac{1}{3})^4 (\frac{2}{3})^2 + \binom{6}{5} (\frac{1}{3})^5 (\frac{2}{3}) + \binom{6}{6} (\frac{1}{3})^6 (\frac{2}{3})^0 = \frac{73}{729} \quad .$$

Ans. to 5.6: The probability that team B scored 4 goals before team A's 3^{rd} goal is $\frac{73}{729}$.