Dr. Z.'s Probability Lecture 4 Handout: Conditional Probability and Independence; Bayes's Formula

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Important Definition

If $P(F) > 0$, the **conditional probability** of the event E given that we know that the event F happened, denoted by $P(E|F)$, is given by the formula

$$P(E|F) := \frac{P(EF)}{P(F)} \quad .$$

Explanation: If we know that F happened then the "sample space" becomes F, and the "new event" is EF, the set of atomic events common to E and F (those that do not belong to F are no longer relevant), so the definition makes sense.

Problem 4.1: Suppose that you tossed a fair coin 10 times and you got more than 6 heads. What is the probability that you get less than 9 heads.

Sol. to Problem 4.1:

$$P(F) = \frac{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}} \quad .$$

The event EF is getting either 7 heads or 8 heads so

$$P(EF) = \frac{\binom{10}{7} + \binom{10}{8}}{2^{10}} \quad .$$

Hence, the desired conditional probability is

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\binom{10}{7} + \binom{10}{8}}{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}} = \frac{15}{16} \quad . \quad .$$

Problem 4.2: Assuming that it is equally likely for a child to be a boy or a girl answer the following questions.

(i) If you know that a family has four children, what is the probability that they are all boys?

(ii) If you know that a family has four children, and that they have at least one boy, what is the probability that they are all boys?

(iii) If you know that a family has four children (all different ages), and that the eldest is a boy, what is the probability that they are all boys?

(iv) If you know that a family has four children, and that they have at least two boys, what is the probability that they are all boys?

Sol. to 4.2:

(i) Here there is no conditional probability, and the probability is $\frac{1}{16}$, since $E = \{bbbb\}$ and the sample space has 2^4 elements.

(ii) Now $E = \{bbbb\}$ but

$$F = \{b, g\}^4 \backslash \{gggg\}$$

so $P(F) = \frac{15}{16}$. Of course $EF = E$ (E is a subset of F), and so $P(EF) = P(E) = \frac{1}{16}$, hence the desired conditional probability is $\frac{1}{15}$.

(iii) Now F has eight elements $\{bbbb, \ldots, bggg\}$ (all the combinations with b at the beginning), and once again $EF = \{bbbb\}$, so $P(E|F) = \frac{1}{8}$. Of course, a more direct way to get it is just considering the sample space of the three younger children.

(iv) Now

$$F = \{bbgg, bgbg, bggb, gbbg, gbgb, ggbb, bbbg, bbgb, bgbb, gbbbb, bbbb\} \quad ,$$

so $P(F) = \frac{11}{16}$. Once again $P(EF) = \frac{1}{16}$, so the desired probability is $\frac{1}{11}$.

Important formula (the multiplication rule)

For **two** events:

$$P(E_1 E_2) = P(E_1) P(E_2|E_1) \quad .$$

Note: This follows immediately from the formula for conditional probability: $P(E|F) := \frac{P(EF)}{P(F)}$.

For **three** events:

$$P(E_1 E_2 E_3) = P(E_1) P(E_2|E_1) P(E_3|E_1 E_2) \quad ,$$

(and analogously for more events).

Problem 4.3: Nancy wants to be a doctor, but first she must graduate from high school, then get into college, then graduate from college, then get into medical school, then graduate from medical school, and then pass the medical board exams.

It is known that

- the probability of graduating from high school is 0.8
- the probability of a high-school graduate being admitted to some college is 0.7
- the probability of someone who starts college to graduate is 0.6
- the probability of a college graduate to get accepted to a medical school is 0.2

• the probability of a person who was accepted to medical school to graduate it with a passing grade is 0.6

• the probability of a person who was graduated successfully from medical school to pass the medical board exams is 0.8.

Assuming that Nancy is completely random, what is her chance to become a doctor one day?

Ans. to 4.3: $0.8 \cdot 0.7 \cdot 0.6 \cdot 0.2 \cdot 0.6 \cdot 0.8 = 0.032256$, so Nancy's chance to realize her ambition is about %3.22.

Problem 4.4: In a certain class, the only sports played are soccer and basketball.

• the probability that you only play soccer is 0.3

• the probability that you only play basketball is 0.2

• the probability that you play both soccer and basketball is 0.1

Find

(i) The conditional probability that you also play basketball if it is known that you play soccer.

(ii) The conditional probability that you play soccer if it is known that you play basketball.

(iii) The conditional probability that you don't play soccer if it is known that you don't play basketball.

(iv) The conditional probability that you don't play basketball if it is known that you don't play soccer.

Sol. to 4.4: The very **first** step, before we can go on, is to figure the *missing piece*, the probability of playing neither soccer nor basketball. The data of the problem can be abbreviated (where S means soccer and B means basketball).

$$P(B^cS) = 0.3 \quad , \quad P(BS^c) = 0.2 \quad , \quad P(BS) = 0.1 \quad .$$

We need, first, to figure out $P(B^cS^c)$. Since $\{B^cS^c, B^cS, BS^c, BS\}$ covers **all** scenarios, and the probabilities must add-up to one, we have $P(B^cS^c) = 1 - (0.3 + 0.2 + 0.1) = 0.4$. So the extended data is

$$P(B^cS^c) = 0.4 \quad , \quad P(B^cS) = 0.3 \quad , \quad P(BS^c) = 0.2 \quad , \quad P(BS) = 0.1 \quad .$$

Before going on, it is convenient to compute $P(S), P(B), P(S^c), P(B^c)$.

$$P(S) = P(B^cS) + P(BS) = 0.3 + 0.1 = 0.4 \quad , \quad hence \quad , \quad P(S^c) = 1 - 0.4 = 0.6 \quad ;$$

$$P(B) = P(BS^c) + P(BS) = 0.2 + 0.1 = 0.3 \quad , \quad hence \quad , \quad P(B^c) = 1 - 0.3 = 0.7 \quad .$$

We can now answer all the questions.

(i) $P(B|S) = P(BS)/P(S) = 0.1/(0.4) = \frac{1}{4}$;

(ii) $P(S|B) = P(BS)/P(B) = 0.1/(0.3) = \frac{1}{3}$;

(iii) $P(S^c|B^c) = P(B^cS^c)/P(B^c) = 0.4/(0.7) = \frac{4}{7}$;

(iv) $P(B^c|S^c) = P(B^cS^c)/P(S^c) = 0.4/(0.6) = \frac{2}{3}$.

Ans, to 4.4: (i) $\frac{1}{4}$, (ii) $\frac{1}{3}$ (iii) $\frac{4}{7}$ (iv) $\frac{2}{3}$.

Problem 4.5: In a certain string-music school, some people play the violin, some the cello, and some neither (because they play viola, or guitar, or bass, or whatever).

• For each of the two instruments (violin and cello), the probability is 0.1 that a student only plays that instrument.

• The probability that a student plays both instruments given that he plays cello is $\frac{1}{3}$.

What is the conditional probability that a student plays none of the instruments, given that he does not play the violin.

Sol. to 4.5: From the data we know (V=violin, C=cello)

$$P(CV^c) = 0.1 \quad , \quad P(C^cV) = 0.1 \quad .$$

We need algebra to go on. Let $P(CV) = x$, then since $\{C^cV^c, C^cV, CV^c, CV\}$ covers everything, we have

$$P(C^cV^c) = 1 - (x + 0.1 + 0.1) = 0.8 - x.$$

Hence our "atomic" data is as follows

$$P(C^cV^c) = 0.8 - x \quad , \quad P(CV^c) = 0.1 \quad , \quad P(C^cV) = 0.1 \quad P(CV) = x \quad .$$

The next thing is to find the value of x. From the problem we know that

$$P(V|C) = \frac{1}{3} \quad .$$

But

$$P(V|C) = P(CV)/P(C) \quad ,$$

so in order to use this fact, we need to find $P(C)$. Since $P(C) = P(CV^c) + P(CV)$, we have, from the above table $P(C) = 0.1 + x$. Hence we have

$$\frac{x}{0.1 + x} = \frac{1}{3} \quad ,$$

leading to $3x = 0.1 + x$, and so $2x = 0.1$ and $x = 0.05$. Now we can make the above table more concrete

$$P(C^cV^c) = 0.75 \quad , \quad P(CV^c) = 0.1 \quad , \quad P(C^cV) = 0.1 \quad P(CV) = 0.05 \quad .$$

We need $P(C^c|V^c)$. By the definition of conditional probability this is $P(C^cV^c)/P(V^c) = (0.75)/P(V^c)$. Now $P(V^c) = P(C^cV^c) + P(CV^c) = 0.75 + 0.1 = 0.85$, so we have, at long last that the desired conditional probability is $0.75/0.85 = \frac{15}{17}$.

Ans. to 4.5: The conditional probability that a student plays none of the instruments, given that he does not play the violin is $\frac{15}{17} = 0.882352941\ldots$, about %88.24.

Important Formula: (Bayes's Formula for two Scenarios)

Suppose that there are only two scenarios F_1 and F_2, (so $P(F_1) + P(F_2) = 1$).

You know that, for some event E that you are interested in

- If scenario F_1 happened the probability of E happening is $P(E|F_1)$

- If scenario F_2 happened the probability of E happening is $P(E|F_2)$

Then of course

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) \quad .$$

Now you are told that E **did** happen, but you don't know whether it came from F_1 or F_2, Bayes's formula tells you that

$$P(F_1|E) = \frac{P(E|F_1)P(F_1)}{P(E)} \quad , \quad P(F_2|E) = \frac{P(E|F_2)P(F_2)}{P(E)} \quad .$$

Note: The formula makes sense, since the first part of the formula $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$ is the contribution to $P(E)$ due to stuff from F_1 and the second is due to stuff from F_2 so each gets its proportional share.

Problem 4.6: A certain pregnancy test, that is equally likely to be taken by women who are pregnant and women who are not pregnant (so the **prior** probability is 0.5 for each) is correct with probability 0.98 if you are pregnant (i.e. the probability for a false negative is 0.02), but is only 0.93 correct if you are not pregnant (i.e. the probability for a false positive is 0.07). If someone took the pregnancy test, and it came out positive what is the probability that she is actually pregnant?,

Sol. to 4.6: Here E is the event that the test said that you were pregnant, F_1 is the event that you are pregnant, and F_2 is the event that your are not pregnant.

The data is

$$P(E|F_1) = 0.98 \quad , P(E|F_2) = 0.07 \quad .$$

Since we are assuming that $P(F_1) = 0.5$ and $P(F_2) = 0.5$, we hava:

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) = 0.98 \cdot 0.5 + 0.07 \cdot 0.5 = 0.525 \quad .$$

Hence the *a posteriori* probability that the woman is actually pregnant is:

$$P(F_1|E) = \frac{0.98 \cdot .5}{0.98 \cdot .5 + 0.07 \cdot .5} = 0.93333\ldots \quad .$$

Ans. to 4.6: The probability that a woman who was tested positive in the pregnancy test is in fact pregnant is $\%93.333\ldots$.

Important Formula: (Bayes's Formula for three Scenarios)

Suppose that there are only three scenarios F_1, F_2, and F_3 (so $P(F_1) + P(F_2) + P(F_3) = 1$).

You know that, for some event E that you are interested in

• If scenario F_1 happened the probability of E happening is $P(E|F_1)$

• If scenario F_2 happened the probability of E happening is $P(E|F_2)$

• If scenario F_3 happened the probability of E happening is $P(E|F_3)$

Then of course

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + P(E|F_3)P(F_3) \quad .$$

Now you are told that E **did** happen, but you don't know whether it came from F_1 or F_2, or F_3. Bayes's formula tells you that

$$P(F_1|E) = \frac{P(E|F_1)P(F_1)}{P(E)} \quad , \quad P(F_2|E) = \frac{P(E|F_2)P(F_2)}{P(E)} \quad , \quad P(F_3|E) = \frac{P(E|F_3)P(F_3)}{P(E)} \quad .$$

Problem 4.7 A certain hard class has students from three different departments, math, physics, and biology. It is known that

(i) 30% of the students are from the math department

(ii) 40% of the students are from the physics department

(iii) The remaining students are from the biology department

(iv) 30% of the students from the math department got an A.

(v) 20% of the students from the physics department got an A

(vi) 10% of the students from the biology department got an A.

If a student got an A, what is the probability that he or she is from the biology department?

Sol. to 4.7 If E is the event of getting an A, we have, and F_1, F_2, F_3 are the events of coming from the math, physics, and biology departments respectively, we have

$$P(F_1) = 0.3 \quad , \quad P(F_2) = 0.4 \quad , \quad P(F_3) = 0.3.$$

$$P(E|F_1) = 0.3 \quad , \quad P(E|F_2) = 0.2 \quad , \quad P(E|F_3) = 0.1 \quad .$$

Hence

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + P(E|F_3)P(F_3) = 0.3 \cdot 0.3 + 0.4 \cdot 0.2 + 0.1 \cdot 0.3 \quad ,$$

and the probability that an A student is from the biology department is

$$P(F_3|E) = \frac{0.1 \cdot 0.3}{0.3 \cdot 0.3 + 0.2 \cdot 0.4 + 0.1 \cdot 0.3} = \frac{0.03}{0.09 + 0.08 + 0.03} = \frac{3}{20} = 0.15 \quad .$$

Ans. to 4.7: The probability that an A student came from the biology department is %15.

Problem 4.8: The probability that a randomly chosen dog is a biter is 0.1. Dogs who are biters are twice as likely to be barkers as those who do not bite.

What is the conditional probability that a randomly chosen dog is a biter, given that it is a barker?

Sol. of Problem 4.8: $P(Biter) = 0.1$, $P(NonBiter) = 0, 9$. Let $P(Barker|NonBiter) = p$ (we have no clue what is it, but we don't need to know it). Then $P(Barker|Biter) = 2p$.

By Bayes's formula (for two scenarios)

$$P(Biter|Barker) = \frac{P(Barker|Biter) \cdot P(Biter)}{P(Barker|Biter) \cdot P(Biter) + P(Barker|NonBiter) \cdot P(NonBiter)}$$

$$= \frac{2p \cdot 0.1}{2p \cdot 0.1 + p \cdot 0.9} = \frac{p(0.2)}{p(2 \cdot 0.1 + 0.9)} = \frac{0.2}{0.2 + 0.9} = \frac{2}{11} \quad .$$

(Note that p canceled out!).

Ans. to 4.8: The conditional probability that a randomly chosen dog is a biter, given that it is a barker is $\frac{2}{11}$.

Problem 4.9: In a certain country, it was found out that

• Ten percent of the population drink but do not smoke, twenty percent of the population smoke but do not drink, and thirty percent smoke and drink.

• The probability of a person who smokes but does not drink to die before the age of seventy is three times the probability of a person who neither smokes nor drinks .

• The probability of a person who drinks but does not smoke to die before the age of seventy is twice times the probability of a person who neither smokes nor drinks .

• The probability of a person who drinks and smokes to die before the age of seventy is four times the probability of a person who neither smokes nor drinks.

If you pass a funeral, and you are told that the deceased died at an age younger than seventy, what is the probability that he neither smoked nor drank?

Sol. to 4.9: The prior probabilities are

$$P(DS^c) = 0.1 \quad , \quad P(D^cS) = 0.2 \quad , \quad P(DS) = 0.3 \quad , \quad P(D^cS^c) = 0.4 \quad .$$

The last fact was implied by the first three, since all the prior probabilities have to add-up to one). Let p be the probability that someone who neither smokes nor drinks would die before the age of seventy be p. Then, if E is the event "*die before the age of seventy*", we have,

$$P(E|D^cS^c) = p \quad , \quad P(E|D^cS) = 3p \quad , \quad P(E|DS^c) = 2p \quad , \quad P(E|DS) = 4p \quad ,$$

Hence

$$P(E) = P(E|D^cS^c) \cdot P(D^cS^c) + P(E|D^cS) \cdot P(D^cS) + P(E|DS^c) \cdot P(DS^c) + P(E|DS) \cdot P(DS)$$

$$p \cdot 0.4 + (3p) \cdot 0.2 + (2p) \cdot 0.1 + (4p) \cdot 0.3 = 2.4p \quad .$$

and, finally, the desired probability is

$$P(D^cS^c|E) = \frac{P(E|D^cS^c) \cdot P(D^cS^c)}{P(E)} = \frac{p \cdot 0.4}{2.4p} = \frac{1}{6}$$

Ans. to 4.9: The probability that the deceased neither smoked nor drank is $\frac{1}{6}$.