Dr. Z.'s Probability Lecture 3 Handout: Sample spaces having equally likely outcomes

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As we have already noted , if you have a finite set, S, of *atomic events*, called the *Sample Space*, or the *Universal Set*, and each atomic event is equally likely, then, of course, the probability of any one atomic event is $\frac{1}{|S|}$. If you have an event E (any subset of the sample space), then

$$P(E) = \frac{Number \quad of \quad Outcomes \quad in \quad E}{Number \quad of \quad Outcomes \quad in \quad S} = \frac{|E|}{|S|} \quad .$$

Problem 3.1: What is the probability that if you toss a fair coin 12 times, you would get exactly 5 Tails?

Sol. to 3.1: Here S is the set $\{h,t\}^{12}$ consisting of all sequences of length 12 consisting of h's and t's. So $|S| = 2^{12}$. E is the subset of such strings with exactly 5 tails, whose number is $\binom{12}{5}$, so the desired probability is

$$\frac{\binom{12}{5}}{2^{12}} \quad .$$

Problem 3.2: What is the probability that if you roll a fair die 10 times, you will not get any ones.

Sol. to 3.2: The sample space is $\{1,2,3,4,5,6\}^{10}$, the set of all vectors of length 10 whose components are drawn from the set $\{1,2,3,4,5,6\}$, so $|S| = 6^{10}$. The set E consists of the set $\{2,3,4,5,6\}^{10}$ the set of all vectors of length 10 whose components are drawn from the set $\{2,3,4,5,6\}$, so $|E| = 5^{10}$. The desired probability is

$$\frac{5^{10}}{6^{10}} \quad .$$

Problem 3.3: What is the probability that if you roll a fair die n times, you will not get a multiple of 3 (i.e. you never get 3 and never get 6)?

Sol. to 3.3: The sample space is $\{1,2,3,4,5,6\}^n$, the set of all vectors of length n whose components are drawn from the set $\{1,2,3,4,5,6\}$, so $|S| = 6^n$. The set E consists of the set $\{1,2,4,5\}^n$ the set of all vectors of length n whose components are drawn from the set $\{1,2,4,5\}$, so $|E| = 4^n$. The desired probability is

$$\frac{4^n}{6^n} = \frac{2^n}{3^n} \quad .$$

Problem 3.4 There are 20 boys and 20 girls, and they have to form teams of two for a math competition. If each arrangement is equally likely, what is the probability that every team is mixed, i.e. each of the 20 teams-of-two consists of one boys and one girl.

Sol. of 3.4: The set S consists of all possible ways of forming 20 pairs out of the 40 students. If the teams were numbered, their number is the multinomial coefficients $M(2,2,\ldots,2)$, where there are 20 2's. This number is

$$\frac{40!}{2!^{20}} = \frac{40!}{2^{20}} \quad .$$

Since the teams are not numbered, we have

$$|S| = \frac{\frac{40!}{2!^{20}}}{20!} = \frac{40!}{2^{20}20!} \quad .$$

The set E has 20! elements since this is the number of ways of matching girls to boys, so that every girl is matched to exactly one boy (and vice versa). So $|E| = 20!$. Hence the desired probability is

$$\frac{|E|}{|S|} = \frac{20!}{\frac{40!}{2^{20}20!}} = \frac{2^{20}20!^2}{40!} = 0.00000760683645\ldots \quad .$$

Problem 3.5: What is the probability that among 23 people, at least two of them have the same birthday?

(Assuming that all 365 days are equally likely, and no one is born on Feb. 29)

Sol. of 3.5: It is easier to find the probability that all the birthdays are different!

The sample space, S is the set of vectors of length 23 whose components are days of the year, i.e. $\{1,2,\ldots,365\}^{23}$, so $|S| = 365^{23}$.

The set E is the set of vectors

$$(x_1, x_2, \ldots, x_{23}) \quad ,$$

whose components belong to $\{1,\ldots,365\}$, but now we have

$$x_2 \neq x_1 \quad , \quad x_3 \notin \{x_1, x_2\} \quad , \ldots , \quad x_{23} \notin \{x_1, x_2, \ldots x_{22}\} \quad .$$

There are 365 choices for x_1; having chosen x_1, there are 364 choices for x_2; having chosen x_1, and x_2 there are 363 choices for x_3, ... etc. So

$$|E| = 365 \cdot 364 \cdot 363 \cdots (365 - 23 + 1) \quad .$$

Hence

$$P(E) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - 23 + 1)}{365^{23}} = 1 \cdot (1 - 1/365) \cdot (1 - 2/365) \cdots (1 - 22/365)$$
$$= 0.492702765676 \quad .$$

This is the probability that all birthdays are **different**. Hence the probability that there are at least two people with the same birthday is $1 - 0.492702765676 = 0.50729723432\ldots$, a little more than %50.

Problem 3.6: If you have a box (where you can't see the inside) consisting of 20 red balls, 30 green balls, 40 yellow balls, and 20 blue balls. and you draw 16 balls (without replacing them). What is the probability that you would pick 3 red balls, 4 green balls, 5 yellow balls, and 4 blue balls.

Sol. to 3.6: S is the set of subsets of size 16 of the set of balls, so $|S| = \binom{110}{16}$. E is the Cartesian product of

- 3-element subsets of the set of 20 red balls, whose number of elements is $\binom{20}{3}$
- 4-element subsets of the set of 30 green balls, whose number of elements is $\binom{30}{4}$
- 5-element subsets of the set of 40 green balls, whose number of elements is $\binom{40}{5}$
- 4-element subsets of the set of 20 green balls, whose number of elements is $\binom{20}{4}$

So

$$|E| = \binom{20}{3} \cdot \binom{30}{4} \cdot \binom{40}{5} \cdot \binom{20}{4} \quad ,$$

so the desired probability is

$$\frac{|E|}{|S|} = \frac{\binom{20}{3} \cdot \binom{30}{4} \cdot \binom{40}{5} \cdot \binom{20}{4}}{\binom{110}{16}} \quad .$$