Dr. Z.'s Probability Lecture 20 Handout: Conditional Expectation

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Important Reminder (from Lecture 16)

If X and Y are discrete random variables with joint mass function $p(x, y)$, the

conditional probability mass function of X **given that** $Y = y$, denoted by $p_{X|Y}(x|y)$

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} \quad ,$$

where, as usual, $p_Y(y)$ is the **marginal distribution** of Y

$$p_Y(y) = \sum_x p(x, y) \quad .$$

Important Definition: If X and Y are discrete random variables, then the **conditional expectation** of X given that $Y = y$ (for all y for which $p_Y(y) > 0$) is

$$E[X \,|\, Y = y] \;=\; \sum_x x \, p_{X|Y}(x|y) \quad .$$

Another way of putting it is

$$E[X \,|\, Y = y] \;=\; \frac{\sum_x x \, p(x, y)}{\sum_x p(x, y)} \quad .$$

Of course, it only depends on y.

Problem 20.1: In a certain community the maximum number of boys is 5 and the maximum number of girls is 5.

It is found that the probability density function

$$p(i, j) = Pr(NumberOfBoys = i, NumberOfGirls = j) = \frac{c(1 + i\,j)}{2 + i\,j} \quad , \quad 0 \leq i \leq 5 \quad , \quad 0 \leq j \leq 5 \quad ,$$

for some constant c (that would make it a legal probability distribution).

(i) Find the expected number of girls in families with i boys for $i = 0, 1, 2, 3, 4, 5$

(ii) Find the expected number of boys in families with j girls for $j = 0, 1, 2, 3, 4, 5$

Express your answer as Maple commands. Do not evaluate.

Sol. to 20.1: Let B be the number of Boys and G the number of girls.

Sol. to 20.1(a): For $0 \leq i \leq 5$, we have

```
add(j*(1+i*j)/(2+i*j),j=0..5)/add((1+i*j)/(2+i*j),j=0..5);
```

Sol. to 20.1(b): For $0 \leq j \leq 5$, we have

```
add(i*(1+i*j)/(2+i*j),i=0..5)/add((1+i*j)/(2+i*j),i=0..5);
```

Note: c cancels out!

Conditional Expectation in General: Sometimes we condition over a more complicated event. The following problem is an example.

Problem 20.2: In a certain community the maximum number of boys is 7 and the maximum number of girls is 7.

It is found that the probability mass function

$$p(i,j) = Pr(NumberOfBoys = i, NumberOfGirls = j) = \begin{cases} \frac{c(i^3+j^3)}{i^3+j^3+1} \quad , \quad 0 \leq i \leq 7 \quad , \quad 0 \leq j \leq 7 \quad ; \\ 0 \quad , \quad otherwise; \end{cases} ,$$

for some constant c (that would make it a legal probability mass function).

(i) Find the expected number of girls if it is known that there are at least as many girls as boys.

(i) Find the expected number of boys if it is known that there are at least as many girls as boys.

Express your answer as Maple commands. Do not evaluate.

Sol. to 20.2: We have to condition on the event $G \geq B$, whose probability is

```
add(add(c*(i**3+j**3)/(i**3+j**3+1),j=i..7),i=0..7).
```

Sol. to 20.2(i):

```
add(add( j* (i**3+j**3)/(i**3+j**3+1),j=i..7),i=0..7)/
add(add((i**3+j**3)/(i**3+j**3+1),j=i..7),i=0..7)    .
```

Sol. to 20.2(ii):

```
add(add( i* (i**3+j**3)/(i**3+j**3+1),j=i..7),i=0..7)/
add(add((i**3+j**3)/(i**3+j**3+1),j=i..7),i=0..7)    .
```

Important Reminder (from Lecture 16)

If X and Y are continuous random variables with joint density function $f(x,y)$, then the **conditional probability density function of** X **given that** $Y = y$, denoted by $f_{X|Y}(x|y)$, is defined by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad ,$$

where, as usual, $f_Y(y)$ is the **marginal distribution** of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx \quad .$$

Note: Of course the formula $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ only makes sense when the denominator, $f_Y(y)$, is larger than 0. Analogously

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad .$$

Important Definition: If X and Y are continuous random variables with joint density function $f(x,y)$, then The **conditional expectation** of X given that $Y = y$ (for all y for which $f_Y(y) > 0$) is

$$E[X \,|\, Y = y] \; = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \quad .$$

Another way of putting it is

$$E[X \,|\, Y = y] \; = \frac{\int_{-\infty}^{\infty} x\, f(x,y)\, dx}{\int_{-\infty}^{\infty} f(x,y)\, dx} \quad .$$

Of course, it only depends on y.

Similarly, the **conditional expectation** of Y given that $X = x$ (for all x for which $p_X(x) > 0$) is

$$E[Y \,|\, X = x] \; = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy \quad .$$

Another way of putting it is

$$E[Y \,|\, X = x] \; = \frac{\int_{-\infty}^{\infty} y\, f(x,y)\, dy}{\int_{-\infty}^{\infty} f(x,y)\, dy} \quad .$$

Of course, it only depends on x.

Note: Of course, often $f(x,y)$ is only non-zero on a finite set, in which case the limits of integration would be finite.

Problem 20.3: Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{4x+6y}{5} & , \quad if \quad 0 < x < 1\,,\, 0 < y < 1\,; \\ 0 & , \quad otherwise. \end{cases} \quad .$$

Find

(i) $E[X \,|\, Y = y]$

(ii) $E[Y\,|\,X=x]$

Sol. to 20.3(i) First Way: The marginal distribution $f_Y(y)$ is

$$f_Y(y) = \int_0^1 \frac{4x+6y}{5}\,dx \;=\; \frac{6y+2}{5}\,y \quad ,0 \le y \le 1 \quad ,$$

(and 0 otherwise). Hence

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} \;=\; \frac{4x+6y}{2+6y} \quad .$$

So we have

$$E[X\,|\,Y=y] \;=\; \int_{x=0}^1 x\, f_{X|Y}(x,y)\,dx = \frac{1}{2+6y}\cdot \int_0^1 x\,(4x+6y)\,dx \;=\; \frac{4+9\,y}{6\,(3\,y+1)} \quad .$$

Sol. to 20.3(i) Second Way:

$$E[X\,|\,Y=y] \;=\; \frac{\int_0^1 x\, f(x,y)\,dx}{\int_0^1 f(x,y)\,dx} = \frac{\int_0^1 x\,(4x+6y)\,dx}{\int_0^1 (4x+6y)\,dx} = \frac{4+9\,y}{6\,(3\,y+1)} \quad .$$

Note: To get Maple to do it type: `int( x*(4*x+6*y),x=0..1)/ int( (4*x+6*y),x=0..1)` .

For Part **(ii)**, let's only do it the second way, using Maple. Typing

`int( y*(4*x+6*y),y=0..1)/ int( (4*x+6*y),y=0..1)` ; gives $\frac{2\,x+2}{4\,x+3}$.

Problem 20.4: Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{60}{23}(x+y^3) \quad , \quad if \quad 0<y<x<1 \\ 0 \quad , \quad otherwise. \end{cases}$$

for some constant c that would make it a good joint density function.

Find (a) $E[X\,|\,Y=y]$; (b) $E[Y\,|\,X=x]$.

Leave your answers in terms of integrals, and express them as Maple commands. If you have Maple, please compute them.

Sol. to 20.4(a): Since $x>y$ when $f(x,y)$ is non-zero, we have

$$E[X\,|\,Y=y] \;=\; \frac{\int_y^1 x\cdot \frac{60}{23}(x+y^3)\,dx}{\int_y^1 \frac{60}{23}(x+y^3)\,dx} \quad ,$$

and in Maple (60/23 cancels out)

`int( x*(x+y**3),x=y..1)/int( x+y**3,x=y..1) ; ,` that Maple claims is

$$\frac{3\,y^4+3\,y^3+2\,y^2+2\,y+2}{3(2\,y^3+y+1)} \quad .$$

Sol. to 20.4(b): Since $y<x$ when $f(x,y)$ is non-zero, we have

$$E[Y\,|\,X=x] \;=\; \frac{\int_0^x y\cdot\frac{60}{23}\,(x+y^3)\,dy}{\int_0^x \frac{60}{23}\,(x+y^3)\,dy} \quad ,$$

and in Maple (60/23 cancels out)

```
int( y*(x+y**3),y=0..x)/int( x+y**3,y=0..x) ; ,
```

that Maple claims is

$$\frac{2x\left(2\,x^2+5\right)}{5(4+x^2)} \quad .$$

Problem 20.5: Suppose that the joint density of X and Y is given by

$$f(x,y)=\begin{cases}\frac{4}{\pi^2}sin^2(x+y) & , \quad if \quad 0<y<x<\pi\\ 0 \quad , \quad otherwise, \end{cases}$$

Find (a) $E[X\,|\,X+Y<\pi]$; (b) $E[Y\,|\,X+Y<\pi]$.

Leave your answers as Maple commands. If you have Maple, please compute them.

Sol. to 20.5: The event on which we condition is $X+Y<\pi$, whose probability is

$$\int_{\substack{0<y<x<\pi\\ x+y<\pi}} f(x,y)\,dx\,dy$$

The region $\{0\le y\le x\le\pi \quad , \quad x+y<\pi\}$ can be described as

$$\{(x,y)\,|\,y\le x\le \pi-y \quad , \quad 0\le y\le \pi/2 \quad \} \quad .$$

Hence the probability of $X<Y$ is

$$\int_0^{\pi/2}\int_y^{\pi-y} f(x,y)\,dx\,dy \quad .$$

For the numerator for (a) we stick an x in front of the integrand and for the numerator for (b) we stick an y in front of the integrand.

So, for (a)

$$E[X\,|\,X+Y<\pi]=\frac{\int_0^{\pi/2}\int_y^{\pi-y} x\sin^2(x+y)\,dx\,dy}{\int_0^{\pi/2}\int_y^{\pi-y}\sin^2(x+y)\,dx\,dy} \quad .$$

In Maple this is

```
int(int(x*sin(x+y)**2,x=y..Pi-y),y=0..Pi/2)/int(int(sin(x+y)**2,x=y..Pi-y),y=0..Pi/2)
;
```

and Maple says that it is

$$\frac{\pi}{2} - \frac{3}{4\pi} \quad .$$

Similarly, For (b)

$$E[Y \mid X+Y<\pi] = \frac{\int_0^{\pi/2} \int_y^{\pi-y} y \sin^2(x+y)\, dx\, dy}{\int_0^{\pi/2} \int_y^{\pi-y} \sin^2(x+y)\, dx\, dy} \quad .$$

In Maple this is

```
int(int(y*sin(x+y)**2,x=y..Pi-y),y=0..Pi/2)/int(int(sin(x+y)**2,x=y..Pi-y),y=0..Pi/2)
;
```

and Maple says that it is

$$\frac{\pi}{6} - \frac{1}{4\pi} \quad .$$

Note that the answer to (b) is one-third the answer for (a). Also note that the answer is pure number that does not depend on x and y, since the bottom and top are **double integrals**.

Problem 20.6: I have two pets, a cat and a dog, and I am very attached to them, but I like the dog more than the cat. The insurance company sells me two life insurance policies, I buy an insurance policy for the cat that would pay me 3000 dollars if the cat dies within the next ten years. It costs me 50 dollars. I also buy an insurance policy for the dog that would pay me 6000 dollars if the dog dies within the next ten years. It costs me 100 dollars.

• The probability that both the cat and the dog will still be alive in ten years is 0.9 (in which case I get nothing from the insurance company).

• The probability that cat will die but the dog will still be alive in ten years is 0.03 (in which case I get 3000 dollars from the insurance company).

• The probability that dog will die but the cat will be still be alive in ten years is 0.02 (in which case I get 6000 dollars from the insurance company).

After deducting the premiums (the cost of the policies), what is my expected gain, if it is known that the cat is alive after ten years.

Sol. to 20.6: The probability that the cat is alive after ten years is $0.9 + 0.02 = 0.92$ (that is our denominator).

If the dog is also alive, I get nothing.

If the dog dies, I would get 6000 dollars. So the conditional expected pyament from the Insruance company is

$$\frac{0.9 \cdot 0 \ + \ 0.02 \cdot 6000}{0.9 \ + \ 0.02} = 130.4347826 \quad .$$

No matter what, I paid $50 + 100 = 150$ dollars for the policies, so the conditional expected **net** gain is

$$130.4347826 - 150 \ = \ -19.5652174\ldots \quad .$$

Ans. to 20.6: The expected gain, if it is known that the cat is alive after ten years, is $-19.5652174\ldots$ dollars.

Problem 20.7: You are at the center of a large maze, and you desperately want to get out. There are four paths.

- If you follow the Northern path, you would get out in 10 minutes.

- If you take the Eastern path you would return to the center in 30 minutes.

- If you take the Southern path you would return to the center in 40 minutes.

- If you follow the Western path, you would get out in 50 minutes.

If the probabilities of you choosing the Northern, Eastern, Southern, and Western paths are 0.1, 0.2, 0.3, and 0.4 respectively, what is the expected time, in minutes. until you get out of the maze.

Sol. to 20.7: Let T be the expected time that you would get out, if you start at the middle

We have

$$T = 0.1 \cdot 10 + 0.2 \cdot (T + 30) + 0.3(T + 40) + 0.4 \cdot 50 \quad ,$$

since

- With prob. 0.1 you would get out in 10 minutes (via the Northern path)

- With prob. 0.2 you would be back at the center (via the Eastern path), but you have wasted 30 minutes, and you are back at the starting point (the middle of the maze).

- With prob. 0.3 you would be back at the center (via the Southern path), but you have wasted 40 minutes, and you are back at the starting point (the middle of the maze).

- With prob. 0.4 you would get out in 50 minutes (via the Western path)

Simplifying the right side we get

$$T = 39.0 + 0.5T \quad .$$

Hence $0.5T = 39$ and we get that $T = 39/0.5 = 78$.

Ans. to 20.7: The expected time of getting out of the maze is 78 minutes.