Dr. Z.'s Probability Lecture 2 Handout: What is Probability? Counting vs. Probability

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Important Notation

If A and B are sets then

- $A \cup B$ is the set of elements that are either in A, or in B, or in **both** of them.

- $A \cap B$ is the set of elements that are in **both** A and B.

Note that $A \cap B$ is often abbreviated AB.

The **empty set** is denoted by $\emptyset$.

If A is a subset of some **universal set**, U then A^c (sometimes written $\bar{A}$) is the set of elements that **do not** belong to A.

Similarly $A \cup B \cup C$ is $(A \cup B) \cup C$, the set of elements that belong to at least one of the sets A, B, C, and ABC is the set of elements that belong to all three sets A, B, C. Similarly for more sets.

Problem 2.1: If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{6, 7, 8\}$. Find

$$(i)\ A \cup B \quad (ii)\ AB \quad (iii)\ AC \quad (iv)\ ABC \quad (v)\ AB \cup C \quad .$$

Ans. to 2.1: (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) $\{3, 4\}$ (iii) $\emptyset$ (iv) $\emptyset$ (v) $\{3, 4, 6, 7, 8\}$.

Problem 2.2: If U is the set of integers i such that $1 \leq i \leq 10$, and $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$ Compute

$$(i)\ (A \cup B)^c \quad , \quad (ii)\ A^c B^c$$

Answer to 2.2: (i) $\{7, 9, 10\}$ (ii) $\{7, 9, 10\}$

Note that, in this problem, $(A \cup B)^c$ and $A^c B^c$ happen to be the same. In fact, it is **always** true that $(A \cup B)^c = A^c B^c$, and it is one of the two famous **de Morgan rules**. The other one is $(AB)^c = A^c \cup B^c$.

Important Definition

If you have a finite set of all possibilities, called the **sample space**, U, consisting of *atomic events* and all of them are equally likely, the **probability** of A, $P(A)$, is $|A|/|U|$.

Problem 2.3: By direct counting, find the probability that if you toss a fair coin 3 times you get exactly two Heads.

Sol. to 2.3: The sample space, U, has $2^3 = 8$ elements $\{hhh, hht, \ldots, ttt\}$. The set A is $\{hht, hth, thh\}$, and has 3 elements, so the desired probability is 3/8.

Problem 2.4: By direct counting, find the probability that if you roll a fair pair of dice, at least one of them lands on 1.

Sol. to 2.4: The sample space, U, has $6^2 = 36$ elements $\{([1,1],[1,2],\ldots,[6,6]\}$. The set A is $\{[1,1],[1,2],[1,3],[1,4],[1,5],[1,6],[2,1],[3,1],[4,1],[5,1],[6,1]\}$ and has 11 elements, so the desired probability is $\frac{11}{36}$.

Obvious by Very Important properties of Probability

$P(\emptyset) = 0$, $P(U) = 1$, and if A and B are disjoint subsets of U, then $P(A \cup B) = P(A) + P(B)$.

Warning: The formula $P(A \cup B) = P(A) + P(B)$ is only valid if A and B are disjoint sets.

If A and B are not necessarily disjoint, then we have the

Inclusion-Exclusion Principle For Two Sets (Probability Version)

$$P(A \cup B) = P(A) + P(B) - P(AB) \quad .$$

An equivalent version is (since $P(A^c B^c) = 1 - P(A \cup B)$):

$$P(A^c B^c) = 1 - P(A) - P(B) + P(AB) \quad .$$

The Enumeration versions are analogous.

Inclusion-Exclusion Principle For Two Sets (Counting Version)

$$|A \cup B| = |A| + |B| - |AB| \quad .$$

Equivalently:

$$|A^c B^c| = |U| - |A| - |B| + |AB| \quad .$$

Problem 2.5: In a class of 30 students, 12 are math majors, 15 are CS majors and 7 are double majors of math and CS. How many students are neither math, nor CS majors?

Sol. to 2.5: Using the second version

$$|M^c C^c| = |U| - |M| - |C| + |MC| = 30 - 12 - 15 + 7 = 10 \quad .$$

Ans. to 2.5: There are exactly 10 students that are neither math nor CS majors.

Problem 2.6: In a class of 30 students, 12 are math majors, 7 are double majors of math and CS, and 10 are neither math nor CS. How many students are CS majors?

Sol. to 2.6: Here we need a bit of algebra. First collect data. Let M be the set of math majors and C the set of CS majors.

$$|U| = 30 \quad , \quad |M| = 12 \quad , \quad |MC| = 7 \quad , \quad |M^c C^c| = 10 \quad .$$

We need to find $|C|$.

Thanks to PIE

$$|M^c C^c| = |U| - |M| - |C| + |MC| \quad .$$

Plugging-in the data, we get

$$10 = 30 - 12 - |C| + 7 \quad .$$

Solving for $|C|$ we get

$$|C| = 15 \quad .$$

Ans. to 2.6: There are 15 CS majors.

The Principle of Inclusion-Exclusion (PIE) For Three Sets (Probability Version)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - (P(AB) + P(AC) + P(BC)) + P(ABC) \quad .$$

An equivalent version is (since $P(A^c B^c C^c) = 1 - P(A \cup B \cup C)$) :

$$P(A^c B^c C^c) = 1 - (P(A) + P(B) + P(C)) + (P(AB) + P(AC) + P(BC)) - P(ABC) \quad .$$

The Enumeration version are analogous.

The Principle of Inclusion-Exclusion (PIE) For Three Sets (Counting Version)

$$|A \cup B \cup C| \;=\; |A| + |B| + |C| - (|AB| + |AC| + |BC|) + |ABC| \quad .$$

An equivalent version is since $\quad |A^c B^c C^c| = |U| - |A \cup B \cup C| \quad$:

$$|A^c B^c C^c| \;=\; |U| - (|A| + |B| + |C|) + (|AB| + |AC| + |BC|) - |ABC| \quad .$$

The PIE for any number of sets (events) is similar (see the book, or wikipedia).

Problem 2.7

In a certain population

• The probability of being handsome is 0.2

• The probability of being clever is 0.1

• The probability of being strong is 0.3

• The probability of being handsome and clever is 0.05

• The probability of being handsome and strong is 0.15

• The probability of being clever and strong is 0.05

• The probability of being handsome, clever, and strong is 0.01

What is the probability of being neither handsome, nor clever, nor strong?

Sol. of 2.7. By the Principle of Inclusion and Exclusion (PIE) for three sets,

$$P(H^cC^cS^c) = 1 - (P(H) + P(C) + P(S)) + (P(HC) + P(HS) + P(CS)) - P(HCS)$$

$$= 1 - (0.2 + 0.1 + 0.3) + (0.05 + 0.15 + 0.05) - 0.01 = 0.64 \quad .$$

Ans. to 2.7: The probability of being neither handsome, nor clever, nor strong is 0.64.

Problem 2.8

In a certain population

• The probability of being clever is twice the probability of being handsome

• The probability of being strong exceeds the probability of being clever by 0.05

• The probability of being handsome and clever is one quarter the probability of being handsome

• The probability of being handsome and strong equals the probability of being handsome and clever

• The probability of being clever and strong is 0.05

• The probability of being handsome, clever, and strong is one tenth of the probability of being strong

If the probability of being neither handsome, nor clever, nor strong is 0.05, what is the probability of being handsome?

Sol. of 2.8: Unlike 2.7 that was a straightforward use of PIE, here we need some algebra. If H stands for handsome, C stands for clever, and S stands for strong, let's the desired quantity, $P(H)$ be called x.

We have

- $P(H) = x$;
- $P(C) = 2x$;
- $P(S) = 2x + 0.05$;
- $P(HC) = 0.25x$;
- $P(HS) = 0.25x$;
- $P(CS) = 0.05$;
- $P(HCS) = 0.1P(S) = 0.1(2x + 0.05) = 0.2x + 0.005$.

Entering all this data into PIE, we have

$$P(H^c C^c S^c) = 1 - (x + 2x + 2x + 0.05) + (0.25x + 0.25x + 0.05) - (0.2x + 0.005) = 0.995 - 4.7x$$

But this equals 0.05, according to the problem. So we have to solve the equation

$$0.995 - 4.7x = 0.05 \quad ,$$

yielding

$$x = \frac{0.995 - 0.05}{4.7} = 0.2010638\ldots \quad .$$

Ans. to 2.8: The probability of being handsome is $0.2010638\ldots$.