Dr. Z.'s Probability Lecture 13 Handout: The Exponential Distribution

By Doron Zeilberger

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Important Continuous Random Variable: Exponential Random Variable

The continuous random variable whose prob. density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & if \quad x \geq 0; \\ 0 \quad if \quad x < 0 \,. \end{cases} \quad ,$$

is called an **exponential random variable** with **parameter** λ.

Important Fact: If X is an exponential random variable with parameter λ then $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

Note: What's nice about an exponential distribution is that it is "memory-less". For many natural and sociological or political scenarios, it describes the amount of time until the next "exciting" even will happen. If there was just a war, the chances that the next war will happen in less one year, is the same if there was no war in the last twenty years.

$$P[X > s + t] = P[X > s] \, P[X > t] \quad .$$

Important Fact: The **cumulative distribution function** of an an exponential r.v with parameter λ is $F(x) = 1 - e^{-\lambda x}$. More usefully, $P(X > x) = \overline{F}(x) = 1 - F(x) = e^{-\lambda x}$. In other words, the probability that it will live **longer** than x is $e^{-\lambda x}$.

Problem 13.1: Let X be an exponential distribution with parameter 2. Find

(a) $P(1 < X < 2)$; (b) $P(X > 2)$; (c) E[X] [(d) Var(X) .

Sol. to 13.1: The density function is $f(x) = 2e^{-2x}, 0 \leq x < \infty$.

(a):

$$P(1 < X < 2) = \int_1^2 2e^{-2x} \, dx = -e^{-2x} \Big|_1^2 = -e^{-4} - -e^{-2} = e^{-2} - e^{-4} = 0.1170196443 \quad .$$

(b): $P(X > 2) = \overline{F}(2) = e^{-2 \cdot 2} = e^{-4} = 0.01831563889 \ldots$.

(c): Since $\lambda = 2$, $E[X] = \frac{1}{\lambda} = \frac{1}{2}$.

(d): Since $\lambda = 2$, $Var(X) = \frac{1}{\lambda^2} = \frac{1}{2^2} = \frac{1}{4}$.

Problem 13.2: Suppose that the number of miles that a car can run before its battery dies is exponentially distributed with an average value of 8000 kilometers. If Ellen desires to take a 5000-kilometer trip, what is the probability that she will be able to complete the trip without replacing the battery?

Sol. to 13.2: Since the average is 8000, the parameter is $\lambda = \frac{1}{8000}$, and the density function is

$$f(x) = \frac{1}{8000}\, e^{-x/8000}$$

and more importantly, $\overline{F}(x) = e^{-x/8000}$. The probability that the battery will outlive the trip is $\overline{F}(5000) = e^{-5000/8000} = e^{-0.6} = 0.5488116361\ldots$

Problem 13.3: The lifetime of a gadget costing 300 is exponentially distributed with mean 3 years. The manufacturer agrees to pay a full refund to a buyer if the gadget fails during the first year following the purchase, a two-thirds refund if it fails between the first and second year, and a one-third refund if it fails between the second and third year.

If the manufacturer sells 1000 gadgets, how much should it expect to pay in refunds?

Sol. to 13.3: Since the average is 3, $\lambda = \frac{1}{3}$. The density function is $f(x) = \frac{1}{3}e^{-x/3}$.

$$P(0 < X < 1) = \int_0^1 \frac{1}{3}e^{x/3} = -e^{-x/3}\Big|_0^1 = 1 - e^{-1/3} \quad ,$$

$$P(1 < X < 2) = \int_1^2 \frac{1}{3}e^{x/3} = -e^{-x/3}\Big|_1^2 = e^{-1/3} - e^{-2/3} \quad ,$$

$$P(2 < X < 3) = \int_2^3 \frac{1}{3}e^{x/3} = -e^{-x/3}\Big|_2^3 = e^{-2/3} - e^{-1} \quad .$$

The expected refund for a single gadget is

$$300 \cdot (1 - e^{-1/3}) + 200 \cdot (e^{-1/3} - e^{-2/3}) + 100 \cdot (e^{-2/3} - e^{-1})$$

$$= 300 - 100\,\mathrm{e}^{-1/3} - 100\,\mathrm{e}^{-2/3} - 100\,\mathrm{e}^{-1} = 140.2172129\ldots \quad .$$

Finally, multiplying it by the number of gadgets, 1000, we get that the manufacturer should expect to pay in refunds \$140217.2129 .

Problem 13.4: An auto-insurance policy has a deductible of 2 and a maximum claim payment of 7. Auto loss amounts follow an exponential distribution with mean 3.

Calculate the expected claim payment made of an auto loss.

Note: We make the assumption that the **maximal claim** is 7, so the maximal claim-payment is 5. In other words, we interpret it as (maximum claim) payment rather than maximum (claim payment). In other words, we assume that one must always pay the deductible.

Sol. of 13.4: Since the mean is 3, $\lambda = 1/3$, and so the pdf is $\frac{1}{3}e^{-x/3}$.

The Claim payment function is

$$C(x) = \begin{cases} 0 \quad if \quad 0 < x < 2\,; \\ x-2 \quad if \quad 2 < x < 7\,; \\ 5 \quad if \quad 7 < x < \infty \end{cases} \quad .$$

The expected claim payment is

$$\int_0^\infty C(x)\,\frac{1}{3}e^{-x/3}\,dx = \int_2^7 (x-2)\,\frac{1}{3}e^{-x/3}\,dx + \int_7^\infty 5\,\frac{1}{3}e^{-x/3}\,dx \quad .$$

If you have Maple (or Wolfram Alpha, available on your smartphone), then it is easy, you type:

`int((x-2)/3*exp(-x/3),x=2..7)+int(5/3*exp(-x/3),x=7..infinity);` ,

and get

`3 exp(-2/3) - 3 exp(-7/3)`, that equals 1.249335453....

Alas, if you **don't** have Maple, then we have to do it the old way, by hand.

Let's do the first integral first. By a change of variable $u = x - 2$

$$\int_2^7 (x-2)\,\frac{1}{3}e^{-x/3}\,dx = \frac{1}{3}\int_0^5 u\,e^{-(u+2)/3}\,du = \frac{e^{-2/3}}{3}\int_0^5 ue^{-u/3}\,du \quad .$$

By another change of variable $z = u/3$ so $du = 3dz$ we get that the integral is

$$\frac{e^{-2/3}}{3}\int_0^{5/3} 3ze^{-z}\,3dz = 3e^{-2/3}\int_0^{5/3} ze^{-z}\,dz \quad .$$

By integration by parts

$$\int ze^{-z}\,dz = -(1+z)e^{-z} + C \quad .$$

So the first integral is

$$3e^{-2/3}\int_0^{5/3} ze^{-z}\,dz \;=\; 3e^{-2/3}\left(-(1+z)e^{-z}\Big|_0^{5/3}\right)$$

$$= 3e^{-2/3}\left(-(1+5/3)e^{-5/3} - -(1+0)e^{-0}\right) = e^{-2/3}(3 - 8\,e^{-5/3}) = 3e^{-2/3} - 8e^{-7/3} \quad .$$

The second integral is easier

$$\int_7^\infty 5\,\frac{1}{3}e^{-x/3}\,dx \;=\; \frac{5}{3}\int_7^\infty e^{-x/3}\,dx \;=\; \frac{5}{3}(-3e^{-x/3})\Big|_7^\infty = 5(0 - -e^{-7/3}) = 5e^{-7/3} \quad .$$

Adding these up we get that the expected claim payment is

$$3e^{-2/3} - 8e^{-7/3} + 5e^{-7/3} \;=\; 3(e^{-2/3} - e^{-7/3}) \;=\; 1.249335453\ldots \quad .$$

Ans. to 13.4: The expected claim payment made of an auto loss is $3(e^{-2/3}-e^{-7/3}) = 1.249335453\ldots$.

Problem 13.5 A piece of equipment is being insured against early failure. The time from the purchase until failure of the equipment is exponentially distributed with mean 4 years. The insurance will pay an amount of B if the equipment fails during the first year, and it pays $0.6B$ if failure occurs during the second year, amd it will pay $0.3B$ if the failure occurs during the third or fourth year. If failure occurs after the first four years, no payment will be made.

At what level must B be set if the expected payment made under this insurance is to be 3000?

Sol. to 13.5: [Corrected Nov. 3, 2017, thanks to Jinhua Xu]

The pdf is $f(x) = \frac{1}{4}e^{-x/4}$. The expected payment is

$$B\int_0^1 \frac{1}{4}e^{-x/4}\,dx \,+\, 0.6B\int_1^2 \frac{1}{4}e^{-x/4}\,dx \,+\, 0.3B\int_2^4 \frac{1}{4}e^{-x/4}\,dx$$

$$= B(-e^{-x/4})\Big|_0^1 + 0.6B(-e^{-x/4})\Big|_1^2 + 0.3B(-e^{-x/4})\Big|_2^4$$

$$= B(1-e^{-1/4}) \,+\, 0.6B(e^{-1/4} \,-\, e^{-1/2}) \,+\, 0.3B(-e^{-1}+e^{-1/2})$$

$$=\; B(1-0.4e^{-1/4} - 0.3e^{-1/2} - 0.3e^{-1}) \;=\; 0.7572761812\,B \quad .$$

We need to set B such that

$$0.7572761812\,B = 3000 \quad .$$

So

$$B = \frac{3000}{0.7572761812} = 7572.761812\ldots \quad .$$

Ans. to 13.5: B must be set to 7572.761812 if the expected payment made under this insurance is to be 3000.