Dr. Z.'s Probability Lecture 11 Handout: Continuous Random Variables; their Expectation and Variance

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Important Concept: A **continuous random variable**, X, on a finite or infinite interval (a,b), has the **probability density function** (often abbreviated **density function**), if for every subset B of (a,b), we have

$$P(X \in B) \,=\, \int_B f(x)\,dx \quad .$$

Very Important Special Case: If $a < c < d < b$ then

$$P(c \leq X \leq d) \,=\, \int_c^d f(x)\,dx \quad .$$

Note: For any specific value e, in the interval, $P(X=e)=0$.

Important Concept: The **cumulative distribution function** of a continuous random variable X with density function $f(x)$ on the interval (a,b) is

$$F(x) \,=\, P(X \leq x) = \int_a^x f(t)\,dt \quad .$$

Very Important Facts: $F'(x) = f(x)$. Also $F(a)=0$ and $F(b)=1$.

Problem 11.1: The density function of a continuous random variable, X, is given by

$$f(x) \,=\, \begin{cases} \frac{x^2}{9} & if \quad 0 \leq x \leq 3\,; \\ 0 & otherwise. \end{cases} \quad .$$

(i) What is the probability that X is between 1 and 2?

(ii) What is the probability that X is between 1 and 2, if it is known that it is between 1 and 3.

Sol. to 11.1(i):

$$P(1 \leq X \leq 2) = \int_1^2 \frac{x^2}{9} = \frac{x^3}{27}\Big|_1^2 = \frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27} \quad .$$

Sol. to 11.1(ii): We need the **conditional probability** $P(1 \leq X \leq 2 \,|\, 1 \leq X \leq 3)$.

$$P(1 \leq X \leq 2 \,|\, 1 \leq X \leq 3) = \frac{P(1 \leq X \leq 2)}{P(1 \leq X \leq 3)} \quad .$$

The denominator is

$$\int_1^3 \frac{x^2}{9} = \frac{x^3}{27}\Big|_1^3 = \frac{3^3}{27} - \frac{1^3}{27} = \frac{26}{27} \quad .$$

The numerator was done in part (i), so

$$P(1 \le X \le 2 \,|\, 1 \le X \le 3) = \frac{\frac{7}{27}}{\frac{26}{27}} = \frac{7}{26} \quad .$$

Ans. to 11.1: The probability that X is between 1 and 2 is $\frac{7}{27}$ and the probability that X is between 1 and 2, if it is known that it is between 1 and 3, is $\frac{7}{26}$.

Important Fact: If X is a continuous random variable with density function $f(x)$ defined on a finite or infinite interval (a, b), the **Expectation** of X, $E[X]$, is given by

$$E[X] \,=\, \int_a^b x f(x)\, dx \quad .$$

More generally, for *any*, function g of X, $g(X)$, we have

$$E[g(X)] \,=\, \int_a^b g(x) f(x)\, dx \quad .$$

In particular

$$E[X^2] \,=\, \int_a^b x^2 f(x)\, dx \quad .$$

Note: Once you know $E[X]$ and $E[X^2]$ you can compute the variance by $Var(X) = E[X^2] - E[X]^2$. Of course, you can also use $\int_a^b (x-\mu)^2 f(x)\, dx$, where $\mu = E[X]$, but the former formula is usually more convenient.

Problem 11.2: Let X be the continuous random variable with density function

$$f(x) \,=\, \begin{cases} \frac{c}{x}, & 1 < x < 3\,; \\ 0 & , \, otherwise. \end{cases}$$

for some constant c.

(a) Calculate $E[X]$, (b) Calculate $Var(X)$.

Sol. to 11.2: First we must find c. Since $\int_1^3 f(x)\, dx = 1$, we have

$$1 \,=\, \int_1^3 \frac{c}{x}\, dx = c \, \log x \Big|_1^3 = c\,(\log 3 - \log 1) = c \, \log 3 \quad .$$

Hence $c \log 3 = 1$ and so $c = \frac{1}{\log 3}$, and so

$$f(x) \,=\, \begin{cases} \frac{1}{\log 3}\, \frac{1}{x}, & 1 < x < 3\,; \\ 0 & , \, otherwise. \end{cases}$$

We now have

$$E[X] = \int_1^3 x \cdot \frac{1}{(\log 3)x} = \int_1^3 \frac{1}{\log 3}\, dx = \frac{1}{\log 3}\left(x\Big|_1^3 \right) = \frac{1}{\log 3}(3-1) = \frac{2}{\log 3} \quad .$$

Also

$$E[X^2] = \int_1^3 x^2 \cdot \frac{1}{(\log 3)\, x} = \frac{1}{\log 3} \int_1^3 x\, dx = \frac{1}{\log 3}\left(\frac{x^2}{2}\Big|_1^3 \right)$$

$$= \frac{1}{\log 3}(3^2 - 1^2)/2 = \frac{4}{\log 3} \quad .$$

Finally

$$Var(X) = E[X^2] - E[X]^2 = \frac{4}{\log 3} - (\frac{2}{\log 3})^2 = \frac{4}{\log 3}(1 - \frac{1}{\log 3}) \quad .$$

Ans. to 11.2: $E[X] = \frac{4}{\log 3}$ and $Var(X) = \frac{4}{\log 3}(1 - \frac{1}{\log 3})$.

Problem 11.3: The lifetime of a machine part has continuous distribution on the interval $(0, 10)$, with probability density function f, where $f(x)$ is proportional to $(1+x)^{-3}$. What is the probability that the lifetime of the machine part is between 5 and 7 ?

Sol. to 11.3: The density function is $c(1+x)^{-3}$, for some c. We must first find c.

$$1 = c \int_0^{10} (1+x)^{-3}\, dx = c\frac{1}{-2}(1+x)^{-2}\Big|_0^{10} = -\frac{c}{2}\left(11^{-2} - 1^{-2}\right)$$

$$= \frac{c}{2}(1 - \frac{1}{121}) = c\,\frac{60}{121} \quad .$$

Hence $c = \frac{121}{60}$, and $f(x) = \frac{121}{60}(1+x)^{-3}$ on the interval $(0, 10)$ (and 0 elsewhere.)

Next, we compute

$$P(5 < X < 7) = \int_5^7 f(x)\, dx = \frac{121}{60} \int_5^7 (1+x)^{-3}\, dx = \frac{121}{60} \cdot \frac{1}{-2}(1+x)^{-2}\Big|_5^7$$

$$= \frac{121}{120}\left((1+5)^{-2} - (1+7)^{-2}\right) = \frac{121}{120}\left((6)^{-2} - (8)^{-2}\right) = \frac{121}{120}\left(\frac{1}{36} - \frac{1}{64}\right) = \frac{847}{69120} \quad .$$

Ans. to 11.3: the probability that the lifetime of the machine part is between 5 and 7 is $\frac{847}{69120} = 0.01225405093\ldots$.

Problem 11.4: Damage to a car in a crash is modeled by a random variable with density function

$$f(x) = \begin{cases} c\,(x^3 + x) & , \quad 0 < x < 10\,; \\ 0 & , otherwise. \end{cases}$$

where c is a constant.

A particular car is insured with a deductible of 2. This car was involved in a crash with damage exceeding the deductible. Calculate the probability that the damage exceeded 5.

Sol. to 11.4: We need the conditional expectation $P(5 < X < 10 \,|\, 2 < X < 10)$, hence we do not need to know c since it would cancel out!

$$P(5 < X < 10 \,|\, 2 < X < 10) \;=\; \frac{P(5 < X < 10)}{P(2 < X < 10)} = \frac{\int_5^{10} c\,(x^3+x)\,dx}{\int_2^{10} c\,(x^3+x)\,dx} = \frac{\int_5^{10} (x^3+x)\,dx}{\int_2^{10} (x^3+x)\,dx} \quad .$$

The numerator is

$$\int_5^{10} (x^3+x)\,dx \;=\; \left(\frac{x^4}{4} + \frac{x^2}{2}\right)\Big|_5^{10} = \frac{9525}{4} \quad .$$

The denominator is

$$\int_2^{10} (x^3+x) \;=\; \frac{x^4}{4} + \frac{x^2}{2}\Big|_2^{10} = 2544 \quad .$$

Hence the required conditional probability is

$$\frac{\frac{9525}{4}}{2544} = \frac{3175}{3392} = 0.9360259434\ldots \quad .$$

Ans. to 11.4: the probability that the damage exceeded 5, given that it exceeded the deductible of 2 is $\frac{3175}{3392}$, about %93.602 .

Problem 11.5 The lifetime in hours of a certain gadget is given by

$$f(x) \;=\; \begin{cases} 0, & 0 < x < 10\,; \\ \frac{10}{x^2} & , x > 10\,. \end{cases}$$

What is the probability that exactly 3 out of 7 gadgets will have to be replaced within the first 20 hours?

Sol. to 11.5: This is a **multi-step** problem, where the second part uses the (discrete) Binomial distribution with parameters 7 and p, where p is the probability that a gadget will have to be replaced within the first 20 hours.

We have

$$p = \int_{10}^{20} f(x)\,dx = \int_{10}^{20} 10x^{-2}\,dx = -\frac{10}{x}\Big|_{10}^{20} = -\frac{10}{20} + \frac{10}{10} = \frac{1}{2} \quad .$$

Hence the probability that exactly 3 out of 7 gadgets will have to be replaced within the first 20 hours is

$$\binom{7}{3} \cdot (\frac{1}{2})^3 (1 - \frac{1}{2})^4 = \binom{7}{3} \frac{1}{2^7} = \frac{35}{128} \quad .$$

Ans. to 11.5: The probability that exactly 3 out of 7 gadgets will have to be replaced within the first 20 hours is $\frac{35}{128} = 0.2734375$.

Problem 11.6: The continuous random variable X has probability density function

$$f(x) \; = \; \begin{cases} \frac{x}{4}, & 0 < x < 1\,; \\ \frac{3x^2}{8} & ,1 < x < 2\,; \\ 0 & elsewhere. \end{cases}$$

Let n be a positive integer, find $E[X^n]$.

Sol. to 11.6:

$$E[X^n] = \int_0^2 x^n f(x)\,dx = \int_0^1 x^n \cdot \frac{x}{4}\,dx \,+\, \int_1^2 x^n \cdot \frac{3x^2}{8}\,dx$$

$$= \frac{1}{4}\int_0^1 x^{n+1} + \frac{3}{8}\int_1^2 x^{n+2}\,dx$$

$$\frac{1}{4}\left(\frac{x^{n+2}}{n+2}\Big|_0^1\right) + \frac{3}{8}\left(\frac{x^{n+3}}{n+3}\Big|_1^2\right)$$

$$\frac{1}{4(n+2)} + \frac{3(2^{n+3}-1)}{8(n+3)} \qquad .$$

Ans. to 11.6: $E[X^n] = \frac{1}{4(n+2)} + \frac{3(2^{n+3}-1)}{8(n+3)}$.

Problem 11.7: The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is the square of that of of Company I.

Determine the probability density function of the monthly profit of Company II.

Sol. to 11.7: Let X be the random variable describing the profit of Company I. Its density function is $f(x)$ (from the problem). Let $F(x)$ be the cumulative distribution function.

Let Y be the random variable describing the profit of Company II, and let $g(x)$ be its density function and $G(x)$ be its cumulative distribution function. We have (by definition)

$$G(x) = P(Y \leq x) \quad .$$

But $Y = X^2$, so

$$G(x) = P(Y \leq x) = P(X^2 \leq x) = P(X \leq \sqrt{x}) = F(\sqrt{x}) \quad .$$

We know that $g(x) = G'(x)$ and $F'(x) = f(x)$. Using the chain rule, we have

$$g(x) = G'(x) \; = \; F(\sqrt{x})' = F'(\sqrt{x}) \cdot (x^{1/2})' = f(\sqrt{x}) \cdot (1/2)x^{-1/2} = \frac{f(\sqrt{x})}{2\sqrt{x}} \quad .$$

Ans. to 11.7: The probability density function of the monthly profit of Company II is $\frac{f(\sqrt{x})}{2\sqrt{x}}$.

Problem 11.8: A company agrees to accept the highest of three sealed bids on a certain property. The bids are regarded as three independent random variables with common cumulative distribution function

$$F(x) = x^5 \quad , \quad 0 < x < 1 \quad .$$

What is the expected value of the accepted bid?

Sol. to 11.8: Let Y be the random variable, "the highest bid", and let X_1, X_2, X_3 be the individual bids. Let $G(x)$ be the cumulative distribution function of Y.

By *independence*, we have

$$G(x) = P(Y \le x) = P(X_1 \le x)\, P(X_2 \le x)\, P(X_3 \le x) \quad .$$

But $P(X_1 \le x) = P(X_2 \le x) = P(X_3 \le x) = F(x) = x^5$. Hence

$$G(x) = (x^5)^3 = x^{15} \quad , \quad 0 < x < 1, \quad .$$

Hence, the density function $g(x)$ is:

$$g(x) = G'(x) = (x^{15})' = 15x^{14} \quad .$$

Finally, the expectation of Y, is

$$E[Y] = \int_0^1 x\, g(x)\, dx = \int_0^1 x\, (15\, x^{14})\, dx = \int_0^1 15\, x^{15}\, dx = \frac{15}{16} \left(x^{16} \Big|_0^1 \right)$$

$$= \frac{15}{16}(1^{16} - 0^{16}) = \frac{15}{16} \quad .$$

Ans. to 11.8: the expected value of the accepted bid is $\frac{15}{16}$.