Dr. Z.'s Probability Lecture 1 Handout: Combinatorial Analysis

By Doron Zeilberger

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Important Definition

If A and B are two finite sets (possibly the same), their *Cartesian product*, written $A \times B$, is the set of all pairs:

$$A \times B := \{(a,b) \,|\, a \in A \quad , \quad b \in B\} \quad .$$

Problem 1.1 Find the Cartesian product of $A \times B$ for the following pairs of finite sets

(i) $A = \{1,2,3\} \quad , \quad B = \{a,b,c\} \quad ;$

(ii) $A = \{1,2,3\} \quad , \quad B = \{3,4,5\} \quad ;$

(iii) $A = \{1,2,3\} \quad , \quad B = \{1,2,3\} \quad .$

Ans. to Problem 1.1

(i) $\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\} \quad .$

(ii) $\{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\} \quad .$

(iii) $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} \quad .$

Warning: The members of a Cartesian product are **ordered pairs**, where **order matters!** $(1,3)$ is a **different** creature than $(3,1)$.

Important Notation: the number of elements of a finite set A, is usually written $|A|$ and sometimes $\#A$. The fancy name for the number of elements of A is the **cardinality** of A.

Obvious but VERY important fact: $|A \times B| = |A| \cdot |B|$.

When you have many sets $A_1, A_2, \ldots, A_k$, the Cartesian product $A_1 \times A_2 \times \ldots \times A_k$ is the set of all (ordered!) k-tuples. $(a_1, \ldots, a_k)$ such that $a_1 \in A_1, \ldots, a_k \in A_k$. Also obviously, (but importantly!) $|A_1 \times A_2 \times \ldots \times A_k| = |A_1||A_1| \cdots |A_k|$.

For example, the set of possibilities for tossing a coin one time is $\{h,t\}$, but the set of possibilities for tossing a coin n times is $\{h,t\}^n$ whose number of elements is 2^n.

Note that some (or all) or the $A_1, A_2, \ldots$ may be the same (for example in the case of $\{h,t\}^n$ where all the A_i's are $\{h,t\}$.

Problem 1.2: If you toss a coin 3 times, keeping track of the order, what is the set of outcomes.

Sol. to 1.2: Here the set of possibilities of **one** throw is $\{Head, Tail\}$, abbreviated $\{h, t\}$. So the set is

$$\{h,t\} \times \{h,t\} \times \{h,t\} = \{(h,h,h),(h,h,t),(h,t,h),(h,t,t),(t,h,h),(t,h,t),(t,t,h),(t,t,t)\} \quad .$$

Direct Enumeration by Brute Force

While there are many formulas for enumerating sets with a given set of properties (see below), many times it is more convenient (and often the only way!), to find all the members of the set, and to actually count them.

Useful skills for Backgammon players

When you play Backgammon you throw a pair of dice. While they look the same, in the *eyes of God* they are different, hence there are $6 \cdot 6 = 36$ different possibilities, and the set of all outcomes is

$$\{(1,1),(1,2),\ldots,(6,5),(6,6)\} \quad .$$

Suppose that you are out and your opponent has a piece distance 6 units from the start (and no other piece in her home base). In order to capture it by a **direct hit**, you need at least one of the dice to be a 6, but in addition $(1,5),(2,4),(3,3),(4,2),(5,1)$ would enable you to capture it.

You can examine the above set of 36 potential outcomes one by one, and pick the "good" ones. In this case the set if

$$\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5),(1,5),(2,4),(3,3),(4,2),(5,1)\} \quad .$$

By actual **counting** you see that the number of good possibilities is 16 (later we will see that assuming the dice are *fair*, it means that your probability of capturing is $16/36 = 4/9$).

Problem 1.3: By directly listing the set of 'successful elements' find the number of ways of

(i) Rolling a pair of standard dice such that at least one of them is even.

(ii) Rolling a pair of standard dice such that both are even.

(iii) Rolling a pair of standard dice so that they add up to 10 or at least one of them is even (or both).

(iv) Tossing a coin four times, so that altogether you got 2 Heads and 2 Tails.

(v) Tossing a coin four times, so that altogether you got 2 Heads and 2 Tails but the first toss is a Head.

Ans. to Problem 1.3

(i) 27 (ii) 9 (iii) 28 (iv) 6 (v) 3

Important Formula

The number of ways of picking k *different* elements out of an n-element set, where *order matters*, is

$$n(n-1)\dots(n-k+1) \quad .$$

Problem 1.4: Ten runners compete in the Olympic Finals for the 100-meter dash, the first will get the gold medal, the second the silver medal, and the third the bronze medal. Of course, no one can get more than one medal. In how many ways can it be done?

Ans. to 1.4: $10 \cdot 9 \cdot 8 = 720$.

Important Special Case: The number of ways of ordering n *different* elements is $n! := 1 \cdot 2 \cdots n$.

Note: Any way of ordering $\{1, 2, \dots n\}$ is called a *permutation*.

Important Formula and Notation

The number of ways of picking k *different* elements out of an n-element set, where *order does not matter*, alias the number of k-element subsets of an n-element set, denoted by $\binom{n}{k}$, is

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{1 \cdot 2 \cdots k} = \frac{n!}{k!(n-k)!} \quad .$$

Problem 1.5 A professor decides to give 5 A's to a class of 30 students. In how many ways can she do it.

Ans. to 1.5: $\binom{30}{5} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 142506 \quad .$

Often we have to do **multi-step** problems, combining several formulas or principles. The following problem is an example.

Problem 1.6: A professor has a class of 20 girls and 40 boys. In order that he will not be accused of discrimination, he decides to give 4 A-s to girls and 8 A-s to boys. In how many different ways can he do it?

Sol. to 1.6: The set of outcomes is the Cartesian product of the problem of assigning the A-s to girls and the problem of assigning the A-s to boys (the decisions are completely independent).

- Ans. to Girls problem: $\binom{20}{4}$
- Ans. to Boys problem: $\binom{40}{8}$

Final answer: $\binom{20}{4} \cdot \binom{40}{8} = 372603198825$.

(Note: in tests and quizzes you don't have to evaluate the answer, so leaving it as $\binom{20}{4} \cdot \binom{40}{8}$ is OK.).

Important Formula: If you have k kinds of objects,

• a_1 *identical* objects of the first kind ,

• a_2 *identical* objects of the second kind ,

...

• a_i *identical* objects of the i-th kind ,

...

• a_k *identical* objects of the k-th kind .

The number of ways of arranging them in a line is the **multinomial coefficient**

$$\frac{(a_1 + \ldots + a_k)!}{a_1! \ldots a_k!} \quad .$$

Problem 1.7 The Smith family has 2 pairs of identical twins and one (identical) triplet. No one (except the parents) can tell twins or triplets apart. In how many different-looking ways can they be arranged in a line?

Ans. to 1.7:

$$\frac{(2+2+3)!}{2!2!3!} = 210 \quad .$$

Problem 1.8

The Smith family has 2 pairs of identical twins and one (identical) triplet.

The Jones family has 4 pairs of identical twins and two (identical) triplets.

(a) In how many different-looking ways can they be arranged on one line?

(b) In how many different-looking ways can they be arranged on two rows such that the Smith family stand in the front row, and the Jones family stand in the back row.

Ans. to 1.8(a):

$$\frac{(2+2+3+2+2+2+2+3+3)!}{2!2!3!2!2!2!2!3!3!} \quad .$$

Ans. to 1.8(b):

$$\frac{(2+2+3)!}{2!2!3!} \cdot \frac{(2+2+2+2+3+3)!}{2!2!2!2!3!3!} \quad .$$

Problem 1.9: In how many ways can you rearrange the letters of $ZEILBERGER$.

Sol. to 1.9: There are 3 E-s, 2 R-s, and one of each of the remaining B, G, I, L, Z. Hence the number of ways is

$$\frac{(3+2+1+1+1+1+1)!}{3!2!1!^5} = \frac{10!}{6 \cdot 2} = 302400 \quad .$$

Problem 1.10: Twenty eight items are arranged in a 4-by-7 array as shown

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ A_8 & A_9 & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} \\ A_{15} & A_{16} & A_{17} & A_{18} & A_{19} & A_{20} & A_{21} \\ A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \end{pmatrix} \quad .$$

Calculate the number of ways to form a set of two distinct items such that no two of the selected items are in the same row or in the same column.

Solution of 1.10: First we must **choose** which 2 rows to pick items from. Here order does not matter. The number of ways of doing it is $\binom{4}{2}$. Having done that, we have to pick 2 columns, but now **order matters**. If we picked, say, the first two rows, then the choice $\{A_1, A_9\}$ is **different** than the choice $\{A_2, A_8\}$. So the number of different ways is

$$\binom{4}{2} \cdot (7 \cdot 6) = 252 \quad .$$

Of course, we can start with picking columns, and then pick rows, getting

$$\binom{7}{2} \cdot (4 \cdot 3) = 252 \quad .$$

Both ways are equally correct.