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Some Comments on Rota's Umbral Calculus

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Rota's Umbral Calculus is put in the context of general Fourier analysis. Also, some shortcuts in the proofs are illustrated and a new characterization of sequences of binomial type is given. Finally it is shown that there are few (classical) orthogonal polynomials of binomial type.

PREREQUISITE

G.–C. Rota and co-workers' excellent papers, [A], [B], [C], are assumed. The present paper is simply a collection of footnotes, and certainly it makes little sense to read a footnote without reading the footnotee first.

1. THE CONNECTION WITH CONTINUOUS FOURIER ANALYSIS

Every shift invariant operator on $C^\infty(R)$ is a convolution operator, that is, the Fourier transform of a multiplication by a function (see, for example, Ehrenpreis [3, p. 141]). The inverse Fourier transforms of polynomials are the distributions supported at the origin (Donoghue [2, p. 103]). Thus every shift invariant operator $Q: \mathbf{P} \to \mathbf{P}$ is of the form $p(z) \to [\phi(t)\, p^v(t)]^\wedge$. Since $(1/i)\, D$ corresponds to multiplication by t, it is possible to write $Q = \phi(D)$ which is a special case of the expansion theorem. By E. Borel's theorem (Narashiman [4]) every formal power series is the Taylor series of some C^∞ function. Conversely every C^∞ function gives a formal power series. Thus if $\phi(0) = 0$ we can expand any other C^∞ function $\psi(t)$, formally, in terms of ϕ: $\psi(t) = \sum a_n \phi^n(t)$. Thus $\hat{\psi} = \sum a_n \hat{\phi}^n$, which gives the general expansion theorem.

2. SOME SHORTCUTS MADE POSSIBLE BY USING UMBRAL OPERATORS FROM THE BEGINNING

To every sequence $\{p_n(x)\}$ for which $\deg p_n(x) = n$ there is a linear operator $\mathscr{P}: \mathbf{P} \to \mathbf{P}$ defined by $\mathscr{P}(x^n) = p_n(x)$, $n \in N$.

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DEFINITION. $\mathscr{P}$ is the basic operator for Q if $\{p_n(x)\}$ is the sequence of basic polynomials for Q. In this case we call $\mathscr{P}$ umbral.

In terms of this definition, the definition in [13, p. 688] reads

(1) $\mathscr{P}(x^0) = x^0$,

(2) $\mathscr{P}(x^n)(0) = 0,\ n > 0$,

(3) $Q\mathscr{P} = \mathscr{P}D$, i.e., $Q = \mathscr{P}D\mathscr{P}^{-1}$.

Thus, the operator $\mathscr{P}$ is umbral if and only if $\mathscr{P}D\mathscr{P}^{-1}$ is a delta operator and (1), (2) are satisfied and then $\mathscr{P}$ is a basic operator with respect to $\mathscr{P}D\mathscr{P}^{-1}$. Similarly, it is possible to modify the definition in [B, p. 698] for Sheffer polynomials.

DEFINITION. $\mathscr{S}$ is a Sheffer operator for the delta operator Q if

(1) $\mathscr{S}(1) = C \neq 0$,

(2) $\mathscr{S}D\mathscr{S}^{-1} = Q$.

To illustrate the shortcuts made possible by these definitions, a short proof of Proposition 1 in [B, p. 703] will be given. In the present notation this proposition reads as follows.

PROPOSITION 1. *Let $\mathscr{P}$ be an operator $\mathbf{P} \to \mathbf{P}$ with $\mathscr{P}(1) = 1$, and let A be a delta operator. $\mathscr{P}$ is a Sheffer operator if and only if there exists a sequence $\{s_n\}$ such that*

$$\mathscr{P}^{-1}A\mathscr{P}(x^n) = \sum_{k\geqslant 0} \binom{n}{k} s_{n-k}x^k.$$

First we need

LEMMA 1. *B is shift invariant if and only if there exists a sequence $\{s_n\}$ such that*

$$B(x^n) = \sum_{k\geqslant 0} \binom{n}{k} s_{n-k}x^k.$$

Proof.

$$B(x^n) = \sum \frac{(n)_{n-k}}{(n-k)!} s_{n-k}x^k = \sum \frac{s_{n-k}}{(n-k)!} D^{n-k}(x^n) = \left(\sum a_kD^k\right)(x^n).$$

The lemma follows from the expansion theorem.

LEMMA 2. *Let A be a delta operator. B is shift invariant if and only if $BA = AB$.*

Proof. By the expansion theorem.

Proposition 1 can now be rephrased.

PROPOSITION 1′. *Let A be a delta operator. $\mathscr{P}$ is a Sheffer operator if and only if $\mathscr{P}^{-1}A\mathscr{P}$ is shift invariant.*

Proof. $\mathscr{P}D\mathscr{P}^{-1}$ is shift invariant $\Leftrightarrow^{(\text{Lemma2})}$ $(\mathscr{P}D\mathscr{P}^{-1})\,A = A(\mathscr{P}D\mathscr{P}^{-1}) \Leftrightarrow D\mathscr{P}^{-1}A\mathscr{P} = \mathscr{P}^{-1}A\mathscr{P}D \Leftrightarrow D(\mathscr{P}^{-1}A\mathscr{P}) = (\mathscr{P}^{-1}A\mathscr{P})\,D \Leftrightarrow^{(\text{Lemma2})} \mathscr{P}^{-1}A\mathscr{P}$ is shift invariant.

Since $(\mathscr{P}D\mathscr{P}^{-1})\,(1) = 0$, $(\mathscr{P}D\mathscr{P}^{-1})\,(x) = c \neq 0$, the proposition follows.

3. Umbral Calculus as Fourier Analysis on N

The association of a sequence $\{a_n\}$ with the linear functional $T\colon \mathscr{P} \to \mathbb{C}$ defined by $T(z^n) = a_n$, is no more and no less the Fourier transform in the function space $\mathscr{F}(N) = \{f; f\colon N \to \mathbb{C}\}$. $\mathscr{F}(N)$ is the dual of $\mathscr{F}_0(N) = \{f\colon N \to \mathbb{C};$ support f is finite$\}$. $\hat{\mathscr{F}}_0(N) = \{\sum_0^N a_n z^n$, for some $n\} = \mathbf{P}$ where we put $z = e^{-in\theta}$. Thus it is only natural to define $\hat{\mathscr{F}} = \hat{\mathscr{F}}_0' = (\hat{\mathscr{F}}_0)' = \mathbf{P}'$, as is done in continuous theory (Ehrenpreis [3, p. 8]). For $f \in \mathscr{F}$ one has $\hat{f}(z^n) = \hat{f}(\hat{\delta}_n) = f(\delta_n) = f(n)$, where $\delta_n(n) = 1$; $\delta_n(k) = 0$, $k \neq n$.

4. The Umbral Algebra and Delta Functionals

4.1. The product of linear functionals [C, pp. 101–103] $LM(p(x)) = L_x M_y(p(x+y))$ is the unique product for which $\delta_{x+y} = \delta_x \delta_y$.

4.2. Setting $\mathscr{P}(x^n) = p_n(x)$, the property of $\{p_n(x)\}$ being of binomial type (and $\mathscr{P}$ an umbral operator), can be expressed $\delta_{x+y}\mathscr{P} = (\delta_x\mathscr{P})\,(\delta_y\mathscr{P})$, where $\delta_x u = u(x)$, for every real x and y. Setting $\mathscr{P}(x) = \delta_x\mathscr{P}$, we have a mapping $R \to \mathbb{C}[z]'$ satisfying $\mathscr{P}(x+y) = \mathscr{P}(x)\,\mathscr{P}(y)$ and thus there must be an $L \in \mathbb{C}[z]'$, the infinitesimal generator, such that $\mathscr{P}(x) = \exp(xL)$. Since $\mathscr{P}$ is umbral, so is $\mathscr{P}^{-1}$ (Proposition 1′ with $A = D$) and therefore there exists an L such that $\delta_x\mathscr{P}^{-1} = \exp xL$, which is Theorem 2(b) in [C, p. 106]. Conversely $\exp xL$ satisfies $\exp(x+y)L = (\exp xL)\,(\exp yL)$, which implies that $\mathscr{P}$ given by $\delta_x\mathscr{P} = \exp xL$ is umbral which implies that $\mathscr{P}^{-1}$ is umbral, which implies Theorem 2(a) in [C].

4.3. Note that if $p_n(x) = \mathscr{P}(x^n)$, the conjugate sequence $q_n(x)$ is given by $q_n(x) = \mathscr{P}^{-1}(x^n)$. This proves Theorem 4 in [C, p. 111].

5. Sometimes the Mere Notion of a Linear Functional Can Go a Long Way

All the properties of Laguerre polynomials can be obtained by merely using the notion of linear functionals. It is not necessary that they be a basic sequence to some delta operator. Our approach is to forget that $\{L_n(x)\}$ are polynomials, fix $x = x_0$, and consider the numerical sequence $\{L_n(x_0)\}_{n=0}^{\infty}$. Recall that

$$L_n(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \frac{x^k}{k!}, \tag{5.1}$$

and define $T^x, L^x \in \mathbb{C}[z]'$, $T^x(z^k) = x^k/k!; L^x(z^k) = L_k(x)$ and extend by linearity. By (1.5) we have

$$L^x(z^n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^x(z^k) = T^x\left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k\right) = T^x((1-z)^n).$$

Since (z^n) is a basis for $\mathbb{C}[z]$ we have

$$L^x(u(z)) = T^x(u(1-z)). \tag{5.2}$$

Also,

$$T^{ax}(u(z)) = T^x(u(az)). \tag{5.3}$$

Thus

$$T^x(u(z)) = L^x(u(1-z)). \tag{5.4}$$

Putting $u(z) = z^n$ in (5.4) yields the inverse formula

$$\frac{x^n}{n!} = \sum (-1)^k \binom{n}{k} L_k(x).$$

We also have

$$\begin{aligned} L^{ax}(u(z)) &= T^{ax}(u(1-z)) = T^x(u(1-az)) = L^x(u(1-a(1-z))) \\ &= L^x(u((1-a)+az)). \end{aligned}$$

Thus

$$L^{ax}(u(z)) = L^x(u(1-a+az)). \tag{5.5}$$

Putting $u(z) = z^n$ yields Erdelyi's duplication formula [C, p. 137].

6. A New Criterion for Polynomial Sequences of Binomial Type

As already mentioned in Section 3, every function $f: N \to \mathbb{C}$ has a Fourier transform $\hat{f}: \mathbb{C}[z] \to \mathbb{C}$ defined by $\hat{f}(z^k) = f(k)$ (Rota's umbra). Assume $f: N \to \mathbb{C}$ is a solution of the difference equation with constant coefficients

$$\sum_{\alpha=0}^{N} C_\alpha f(n+\alpha) \equiv 0; \tag{6.1}$$

then $0 = \sum_{\alpha=0}^{N} C_\alpha f(n+\alpha) = \sum_{\alpha=0}^{N} C_\alpha \hat{f}(z^{n+\alpha}) = \hat{f}((\sum_0^N C_\alpha z^\alpha)\, z^n)$, for every n, and setting $P(z) = \sum_0^N C_\alpha z^\alpha$, we have $\hat{f}(P(z)\, u(z)) = 0$, $\forall u \in \mathbb{C}[z]$. Thus f is a solution of $\sum_{\alpha=0}^{N} C_\alpha f(n+\alpha) = 0$ if and only if $\hat{f}$ annihilates the ideal $P(z)\, \mathbb{C}[z]$. Introducing the shift operator $Xf(n) = f(n+1)$, one can write (6.1) as

$$\left(\sum_{\alpha=0}^{N} C_\alpha X^\alpha\right) f \equiv 0.$$

Note that $\widehat{Xf} = z\hat{f}$, where for $T \in \mathbb{C}[z]'$, $zT(u) = T(zu)$. (For discrete functions of several variables and partial difference equations, see Zeilberger [5].)

To consider difference equations with polynomial coefficients we simply note that

$$\widehat{nf}(z^n) = \hat{f}(nz^n) = \hat{f}\left(\left(z\frac{d}{dz}\right)(z^n)\right) = \left(z\frac{d}{dz}\right)\hat{f}(z^n),$$

for every n; so the Fourier transform of multiplication by n, $\hat{n}$, is equal to

$$z\frac{d}{dz}, \qquad \text{where} \qquad \left(z\frac{d}{dz}T\right)(u) = T\left(z\frac{d}{dz}u\right), \qquad T \in \mathbb{C}[z]', \quad u \in \mathbb{C}[z].$$

Thus if $f\colon N \to \mathbb{C}$ is a solution of $P(x) f = (\sum_{\alpha=0}^{N} C_\alpha(n)\, X^\alpha) f = 0$, where the C_α's are polynomials, the Fourier transform of $P(x)$, $\widehat{P(x)}$, is a differential operator with polynomial coefficients and $\hat{f}$ annihilates $\widehat{P(x)}\, \mathbb{C}[z]$.

THEOREM. *Let $\{P_n(x)\}$ be a sequence of polynomials and let $f^x(n) = P_n(x)$, $n \in N$. $\{P_n(x)\}$ is of binomial type if and only if there exists a shift invariant operator $S\colon \mathbb{C}[z] \to \mathbb{C}[z]$ such that $\hat{f}^x[(x - zS)\, u(z)] = 0$, $\forall u \in \mathbb{C}[z]$.*

Proof. Suppose $\hat{f}^x[(x - zS)\, u(z)] = 0$, we have to show that $\hat{f}^{x+y} = \hat{f}^x \hat{f}^y$; we have

$$\begin{aligned} \hat{f}^x \cdot \hat{f}^y([x + y - zS]\, u) &= \hat{f}_z^x \hat{f}_w^y((x+y)\, u(z+w) - (z+w)\,(Su)\,(z+w)) \\ &= \hat{f}_z^x \hat{f}_w^y[(x - zS_z)\, u(z+w) + (y - wS_w)\, u(z+w)] \\ &= 0, \end{aligned}$$

since S is shift invariant. Therefore both $\hat{f}^x \cdot \hat{f}^y$ and $\hat{f}^{x+y}$ annihilate $[x + y - zS]\, \mathbb{C}[z]$, which is easily seen to imply $\hat{f}^{x+y} = \hat{f}^x \cdot \hat{f}^y$.

Conversely, if $\{P_n(x)\}$ is of binomial type

$$\sum_{n=0}^{\infty} \frac{P_n(x)\, t^n}{n!} = \exp[xf(t)].$$

Differentiating with respect to t,

$$\sum_{n=1}^{\infty} \frac{P_n(x)\, t^{n-1}}{(n-1)!} = xf'(t) \exp[xf(t)] = f'(t) \sum_{0}^{\infty} \frac{xP_n(x)}{n!} t^n.$$

Let $[f'(t)]^{-1} = \sum_0^\infty a_m t^m$ (remember that $\gamma'(0) \neq 0$ and so $[f'(t)]^{-1}$ exists), we have

$$\sum_{0}^{\infty} \frac{xP_n(x)}{n!} t^n = \left(\sum_{0}^{\infty} a_m t^m\right) \sum_{k=1}^{\infty} \frac{P_k(x)\, t^{k-1}}{(k-1)!}.$$

Comparing terms we obtain the recurrence equations

$$xP_n(x) = \sum_{k=0}^{n} a_k \left[\frac{n!}{(n-k)!}\right] P_{n-k+1}(x), \qquad (*)$$

which implies

$$\hat{f}^x(xz^n) = \hat{f}^x \left[z \sum_{k=0}^{n} a_k \left[\frac{n!}{(n-k)!}\right] z^{n-k}\right] = \hat{f}^x \left[z \left(\sum_{k=0}^{\infty} a_k D^k\right) z^n\right]$$

and the theorem is true with

$$S = \sum a_k D^k = [f'(D)]^{-1}.$$

Examples

(i) $P_n(x) = x^n$, $P_{n+1}(x) = xP_n(x)$, so $\hat{f}^x((x-z)\,\mathbb{C}[z]) = 0$ and $S = I$.

(ii) $P_n(x) = (x)_n$, $P_{n+1}(x) = (x-n)\, P_n(x)$, $\hat{f}^x([x - z(1 + d/dz)]\,\mathbb{C}[z]) = 0$. Here $S = I + d/dz$.

(iii) Similarly, for $P_n(x) = [x]_n$, $S = -(I + d/dz)$.

(iv) $P_n(x) = L_n^{(-1)}(x)$ satisfy the three-term difference equation

$$xP_n(x) = P_{n+1}(x) + 2nP_n(x) + n(n-1)\, P_{n-1}(x),$$

so

$$\hat{f}^x \left[\left(x - z\left(1 + 2\frac{d}{dz} + \frac{d^2}{dz^2}\right)\right) \mathbb{C}[z]\right] = 0.$$

Here $S = (1 + d/dz)^2$.

(v) In the above examples the shift invariant operators s were differential operators with constant coefficients, of finite order.

We now illustrate an example where S is another shift invariant operator. (Of course every shift invariant operator is an infinite (or finite) differential operator with constant coefficients.)

The exponential polynomials $\{\phi_n(x)\}$ satisfy [A, p. 204; C, p. 139]

$$\phi_{n+1}(x) = x(\phi + 1)^n, \qquad \text{i.e.,} \qquad \phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x),$$

which in our notation is

$$\hat{\phi}^x(z^{n+1}) = x\hat{\phi}^x((1+z)^n), \qquad \text{i.e.,} \qquad \hat{\phi}^x[zu(z) - xu(1+z)] = 0 \qquad \forall u \in \mathbb{C}[z].$$

Replacing $u(z)$ by $u(z-1)$ we obtain

$$\hat{\phi}^x[(x - zE^{-1})\,\mathbb{C}[z]] = 0, \qquad \text{where} \qquad E^{-1}u(z) = u(z-1).$$

Thus $S = E^{-1}$.

7. There are Few Orthogonal Polynomials of Binomial Type

The basic Laguerre polynomials $L_n^{(-1)}(x)$ are both orthogonal (in the classical sense) and of binomial type. We will show that there are not many more such sequences. A sequence of polynomials $\{P_n(x)\}$ is said to be orthogonal, in the classical sense, if there exists an $\mathscr{L}\colon \mathbb{C}[z] \to \mathbb{C}$ such that the inner product is given by $(P(z), Q(z)) = \mathscr{L}(P(z)Q(z))$. Recall (Chihara [1], p. 13]) that every sequence of (monic) orthogonal polynomials satisfies a three-term recurrence relation $xP_n(x) = P_{n+1}(x) + A(n)\,P_n(x) + B(n)\,P_{n-1}(x)$. On the other hand, a sequence of polynomials of binomial type satisfies

$$xP_n(x) = \sum a_k(n)_k\, P_{n-k+1}(x). \tag{*}$$

Thus,

Proposition. *The only orthogonal polynomials of binomial type are those satisfying a recurrence relation of the form*

$$xP_n(x) = P_{n+1}(x) + anP_n(x) + bn(n-1)\,P_{n-1}(x).$$

Note that for $L_n^{(-1)}(x)$, $a = 2$ and $b = 1$.

Note added in proof. S. A. Joni kindly pointed out that the idea of Section 2 was first concieved by A. M. Garsia in *J. Lin. Mult. Algebra* **1** (1973), 47–65. Also M. Ismail informed us that the result of Section 7 goes back to Sheffer.

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