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SIX ETUDES IN GENERATING FUNCTIONS

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Schutzenberger's philosophy of getting algebraic generating functions out of context-free languages is practiced to obtain short proofs of results of Kreweras, Kreweras–Poupard and Kreweras–Moszkowski, concerning the refined enumeration of legal bracketings according to various parameters. We also mention some ramifications to the enumeration of ordered trees. As an "encore", we give a short proof of a formula, conjectured by Kirkman and first proved by Cayley, that counts the number of ways of placing m non-intersecting diagonals in a convex polygon of k sides.

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C.R. CATEGORIES: G.2.1, G.2.2, F.4.3.

0. INTRODUCTION

0.0 Preface

Let L be the set of all *non-empty* words w on the alphabet $\{0,1\}$ such that:

(a) w has as many 0's as 1's.

(b) If $w = l_1 \dots l_i \dots l_{2n}$, then for every $1 \leqq i \leqq 2n$, the first i letters never contain more 1's than 0's.

Thus, $L = \{01, 0101, 0011, 010101, 010011, 001101, 000111, 001011, \dots\}$. The members of L are interchangeably called "legal bracketings" ($0 \to [, 1 \to]$), "Dyck words", "ballot sequences", "subdiagonal walks", "bridges", "sequences of begins and ends in a PASCAL program" etc. etc.

One of the most famous results in enumerative combinatorics is the fact that the number of members of L of length $2n$ is Catalan's number $(2n)!/(n+1)!n!$. In 1956 Narayana [10] discovered a refinement of this fact: he enumerated legal bracketings according to the length and the number of "double zeros". Quite recently, Kreweras and Moszkowski [8] found another parameter that yields the same formula and Kreweras [7] and Kreweras–Poupard [9] found further refinements that generalize the Narayana [10] and the Kreweras–Moszkowski [8] results at the same time. Here I present short proofs of these results using Schutzenberger's powerful methodology of generating generating functions in the context of context-free languages. Gerard Viennot and his "Ecole Bordelaise" (e.g. [2], [17], [18]) have already applied this method extensively.

According to the modern viewpoint, *generating functions* are neither "generating" nor are they "functions". Generating functions that arise in combinatorics are *formal power series* that are *weight enumerators* of sets. Like so many revolutionary concepts, it is hard to attribute this modern approach to generating functions to any one person, but I think that it is fair to say that Schutzenberger (e.g. [13]), in the theory of formal languages, Tutte (e.g. [16]), in graphical enumeration, and Polya ([11]) were among the most prominent in this conceptual revolution. Unfortunately, this brilliant "paradigm shift" is not yet common knowledge. A comprehensive account of this modern approach to generating functions can be found in Goulden and Jackson [3].

0.1 Definitions

For any set S, $\#S$ denotes the number of elements in S. For a word w in L, let us make the following definitions.

i) $n(w) := $ number of 0's in w ($=$ number of 1's $=$ length$(w)/2$).

ii) $h_1(w) := $ number of occurrences of "00" in odd-even positions $= \#\{1 \leqq i \leqq n; l_{2i-1} = l_{2i} = 0\}$.

$h_2(w) := $ number of occurrences of "00" in even-odd positions $= \#\{1 \leqq i \leqq n-1; l_{2i} = l_{2i+1} = 0\}$.

$h(w) := h_1(w) + h_2(w) = $ total number of occurrences of "00" $= \#\{1 \leqq i \leqq 2n-1; l_i = l_{i+1} = 0\}$.

iii) $d(w) := $ number of 0's in even positions $= \#\{1 \leqq i \leqq n; l_{2i} = 0\}$.

iv) $k_1(w) := $ number of occurrences of "001" $= \#\{1 \leqq i \leqq 2n-2; l_i = l_{i+1} = 0$ and $l_{i+2} = 1\}$.

$k_2(w) := $ number of occurrences of "110" $= \#\{1 \leqq i \leqq 2n-2;\ l_i = l_{i+1} = 1$ and $l_{i+2} = 0\}$.

$k(w) := k_1(w) + k_2(w)$.

For example, for $w = 0100011101010011$, we have: $n(w) = 8$; $h_1(w) = h_1(0100011101010011) = 2$; $h_2(w) = h_2(0100011101010011) = 1$; $h(w) = 2 + 1 = 3$; $d(w) = d(0100011101010011) = 2$; $k_1(w) = k_1(0100011101010011) = 2$; $k_2(w) = k_2(0100011101010011) = 1$; $k(w) = 2 + 1 = 3$;

Let:

$$N_0(n, \alpha) = \#\{w \in L; n(w) = n \text{ and } h(w) = \alpha\};$$

$$N_1(n, \alpha) = \#\{w \in L; n(w) = n \text{ and } d(w) = \alpha\};$$

$$N_2(n,\alpha)=\#\{w\in L; n(w)=n \text{ and } k(w)=\alpha\};$$

$$E_0(n,\lambda,\rho,\sigma)=\#\{w\in L; n(w)=n \text{ and } h(w)=\lambda \text{ and } k_1(w)=\rho \text{ and } k_2(w)=\sigma);$$

$$E_1(n,\lambda,\rho,\sigma)=\#\{w\in L; n(w)=n \text{ and } d(w)=\lambda \text{ and } h_1(w)=\rho \text{ and } h_2(w)=\sigma);$$

$$E_2(n,\lambda,\rho,\sigma)=\#\{w\in L; n(w)=n \text{ and } d(w)=\lambda \text{ and } k_1(w)=\rho \text{ and } k_2(w)=\sigma);$$

0.2 Theorems

I will give new proofs of the following results:

THEOREM 1 (see Kreweras [7]) $N_1(n,\alpha)=N_0(n,\alpha)$.

THEOREM 2 (Kreweras–Moszkowski [8]) $N_2(n,\alpha)=N_0(n,\alpha)$.

THEOREM 3 (Narayana [10])

$$N_0(n,\alpha)=\frac{1}{n}\binom{n}{\alpha}\binom{n}{\alpha+1}.$$

THEOREM 4 (Kreweras [7], Section 2) $E_1(n,\lambda,\rho,\sigma)=E_0(n,\lambda,\rho,\sigma)$.

THEOREM 5 (Kreweras [7], Section 3) $E_2(n,\lambda,\rho,\sigma)=E_0(n,\lambda,\rho,\sigma)$.

THEOREM 6 (Kreweras–Poupard [9])

$$E_0(n,\lambda,\rho,\sigma)=\frac{1}{\lambda}\binom{\lambda}{\rho}\binom{\lambda}{\sigma}\binom{n-\lambda-1}{\rho-1}\binom{n-\lambda}{\sigma+1}.$$

0.3 Remarks

META-THEOREM 1 *Theorems* $\{i; 4\leqq i\leqq 6\}$ imply Theorems $\{i; 1\leqq i\leqq 3\}$.

Proof Theorem 4 implies Theorem 1 by summing over ρ and σ. Theorems 4 and 5 imply that $E_1=E_2$, and Theorem 2 follows by summing over λ and $\rho+\sigma=\alpha$. Theorem 3 follows from Theorem 6 by Vandermonde–Chu ([4], p. 59) summations with respect to ρ and σ. □

the same time. Here I present short proofs of these results using Schutzenberger's powerful methodology of generating generating functions in the context of context-free languages. Gerard Viennot and his "Ecole Bordelaise" (e.g. [2], [17], [18]) have already applied this method extensively.

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$h(w) := h_1(w) + h_2(w) = $ total number of occurrences of "00" $= \#\{1 \leqq i \leqq 2n-1; l_i = l_{i+1} = 0\}$.

iii) $d(w) := $ number of 0's in even positions $= \#\{1 \leqq i \leqq n; l_{2i} = 0\}$.

iv) $k_1(w) := $ number of occurrences of "001" $= \#\{1 \leqq i \leqq 2n-2; l_i = l_{i+1} = 0$ and $l_{i+2} = 1\}$.

$k_2(w) := $ number of occurrences of "110" $= \#\{1 \leqq i \leqq 2n-2;\ l_i = l_{i+1} = 1$ and $l_{i+2} = 0\}$.

$k(w) := k_1(w) + k_2(w)$.

For example, for $w = 0100011101010011$, we have: $n(w) = 8$; $h_1(w) = h_1(01\mathbf{00}01110101\mathbf{00}11) = 2$; $h_2(w) = h_2(010\mathbf{00}11101010011) = 1$; $h(w) = 2+1 = 3$; $d(w) = d(010\mathbf{0}011101010\mathbf{0}11) = 2$; $k_1(w) = k_1(010\mathbf{001}1101010\mathbf{001}1) = 2$; $k_2(w) = k_2(010001\mathbf{110}1010011) = 1$; $k(w) = 2+1 = 3$;

Let:

$$N_0(n, \alpha) = \#\{w \in L; n(w) = n \text{ and } h(w) = \alpha\};$$

$$N_1(n, \alpha) = \#\{w \in L; n(w) = n \text{ and } d(w) = \alpha\};$$

It follows that, from a strictly logical viewpoint, if suffices to prove Theorems 4, 5, 6. However, since the point of this article is to illustrate a *method*, I first prove Theorems 1, 2, 3 as a *warm up*.

In Section 2, I have used very routine MACSYMA calculations. These are easily reproducible by anyone who has access to a computer algebra package. For the "record", however, I have submitted a transcript for the referee.

Before going on, I should point out that while the original proofs in [7], [8], [9] are longer than the proofs in this paper, they are not less elegant. I would like to thank Dr Paul Moszkowski for helpful remarks.

1. PROOFS OF THEOREMS 1 AND 2

It is well known and easy to see that every member of L is either the word 01 or else has one of the following forms: $01w$, $0w1$, $0u1v$, where w,u,v are themselves members of L. (Please note that we do *not* include the empty word in L.)

Schematically, this is written as

$$L = 01 \cup 01L \cup 0L1 \cup 0L1L. \tag{1.1}$$

Introduce the following weight on L:

$$W_0(w) = x^{n(w)} t^{h(w)},$$

and define

$$\phi_0(x,t) := W_0(L) \left(:= \sum_{w \in L} W_0(w) \right).$$

$\phi_0 = \phi_0(x,t)$ is a formal power series in x and t. Obviously, $N_0(n,\alpha)$ equals the coefficient of $x^n t^\alpha$ in $\phi_0(x,t)$.

We need

LEMMA 1 *$\phi_0 = \phi_0(x,t)$ satisfies the quadratic equation*

$$xt\phi_0^2 + (x + xt - 1)\phi_0 + x = 0. \tag{1.2}$$

Proof Let us take the weight W_0 on both sides of (1.1):

$$W_0(L) = W_0(01) + W_0(01L) + W_0(0L1) + W_0(0L1L). \tag{1.3}$$

By definition, $W_0(L) = \phi_0$. Also, $W_0(01) = x$. Since $n(01w) = 1 + n(w)$ and $h(01w) = h(w)$, we have $W_0(01w) = x^{n(01w)} t^{h(01w)} = x^{n(w)+1} t^{h(w)} = xW_0(w)$. It follows that $W_0(01L) = xW_0(L) = x\phi_0$. Similarly $W_0(0L1) = xt\phi_0$ (putting a 0 in front of any word of L increases its h by one), and $W_0(0L1L) = xt\phi_0^2$. Putting it all in (1.2) yields

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$$\phi_0 = x + x\phi_0 + xt\phi_0 + xt\phi_0^2,$$

that gives (1.2). □

Solving the quadratic equation (1.2), we get the following explicit formula

$$\phi_0 = \frac{1 - x - xt - \sqrt{1 - 2x - 2xt - 2x^2t + x^2 + x^2t^2}}{2xt}. \tag{1.4}$$

We took the minus sign in front of the radical rather than the plus sign because only the minus sign gives a genuine formal power series.

THEOREM 1 (see Kreweras [7]): $N_1(n,\alpha) = N_0(n,\alpha)$.

Proof Introduce the weight

$$W_1(w) := x^{n(w)} t^{d(w)},$$

(recall that $d(w)$ is the number of occurrences of a 0 in an even position). Let $\phi_1 = \phi_1(x,t)$ be the formal power series $W_1(L)\,(:= \sum_{w \in L} W_1(w))$. Obviously $N_1(n,\alpha)$ is the coefficient of $x^n t^\alpha$ in $\phi_1(x,t)$, and Theorem 1 is equivalent to the statement that the two formal power series ϕ_0 and ϕ_1 are identical.

Let us try to find an equation that is satisfied by ϕ_1. Taking the weight W_1 on both sides of (1.1) gives:

$$W_1(L) = W_1(01) + W_1(01L) + W_1(0L1) + W_1(0L1L). \tag{1.5}$$

For any formal power series f in x and t (in particular for monomials) let $\hat{f}(x,t) := f(xt, 1/t)$.

Now, as before, $W_1(01) = x$ and $W_1(01L) = x\phi_1$. However, inserting a word w inside a $01: w \to 0w1$ reverses the even and odd positions and $d(0w1)$, the number of 0's in *even* positions of $0w1$ equals the number of 0's in *odd* positions of w. This is equal to $n(w) - d(w)$, since altogether there are $n(w)$ 0's in w. The first 0 in $0w1$ does not contribute to the score of d, since it is in an odd position. Thus $d(0w1) = n(w) - d(w)$. Of course $n(0w1) = n(w) + 1$, so

$$W_1(0w1) = x^{n(0w1)} t^{d(0w1)} = x^{n(w)+1} t^{n(w)-d(w)} = x(x^{n(w)} t^{d(w)})\hat{} = x\hat{W}_1(w).$$

Thus $W_1(0L1) = x\hat{\phi}_1$. Similarly, $W_1(0L1L) = x\hat{\phi}_1\phi_1$. Putting it all in (1.5) yields the following functional equation for $\phi_1(x,t)$:

$$\phi_1 = x + x\phi_1 + x\hat{\phi}_1 + x\hat{\phi}_1\phi_1. \tag{1.6}$$

Of course, (1.6) has a unique solution in the ring of formal power series in x and t (in fact in the ring of f.p.s. in x and xt). A purely routine calculation shows that ϕ_0, as given by (1.4), also satisfies (1.6). Thus $\phi_1 = \phi_0$. □

THEOREM 2 (Kreweras–Moszkowski [8]) $N_2(n,\alpha) = N_0(n,\alpha)$.

Proof Introduce the following weight on L:

$$W_2(w)=x^{n(w)}t^{k(w)},$$

(recall that $k(w)$ is the total number of occurrences of "001" and "110"). Define the formal power series ϕ_2 to be $W_2(L):=\sum_{w\in L}W_2(w)$. Of course $N_2(n,\alpha)$ is the coefficient of $x^n t^\alpha$ in $\phi_2(x,t)$ and the theorem is equivalent to showing that the formal power series ϕ_0 and ϕ_2 are identical. Taking the weight W_2 on both sides of (1.1) gives:

$$W_2(L)=W_2(01)+W_2(01L)+W_2(0L1)+W_2(0L1L). \tag{1.7}$$

It is readily seen that:

i) $W_2(01)=x$.

ii) $W_2(01w)=xW_2(w)$, so $W_2(01L)=x\phi_2$. (we note for later that $W_2(01\cup 01L)=x+x\phi_2$, and $W_2(L-01-01L)=\phi_2-x-x\phi_2$).

iii) If w starts with "01" then $k(0w1)=k(w)+1$ (a new "001" is born at the beginning), otherwise $k(0w1)=k(w)$. Always $n(0w1)=n(w)+1$, so:

$$W_2(0w1)=\begin{cases} xtW_2(w) & w\in 01\cup 01L \\ xW_2(w) & w\in L-01-01L \end{cases}.$$

Thus

$$W_2(0L1)=xt(x+x\phi_2)+x(\phi_2-x-x\phi_2).$$

(iv) $k(0u1v)$ is equal to $k(u)+2+k(v)$ if u starts with "01", and to $k(u)+k(v)+1$ otherwise (there is always a brand new "110" formed from the two last letters of the "$0u1$ part" and the first letter of the "v part" of $0u1v$). It follows, just like in (iii), that

$$W_2(0u1v)=\begin{cases} xt^2W_2(u)W_2(v) & u\in 01\cup 01L \\ xtW_2(u)W_2(v) & u\in L-01-01L \end{cases}.$$

Thus,

$$W_2(0L1L)=xt^2(x+x\phi_2)\phi_2+xt(\phi_2-x-\phi_2)\phi_2.$$

Putting it all in (1.7), we get

$$\phi_2=x+x\phi_2+xt(x+x\phi_2)+x(\phi_2-x-x\phi_2)+xt^2(x+x\phi_2)\phi_2+xt(\phi_2-x-x\phi_2)\phi_2.$$

Transposing ϕ_2 from the left to the right and factorizing, yields

$$(xt-x-1)(xt\phi_2^2+(x+xt-1)\phi_2+x)=0,$$

which implies,

$$xt\phi_2^2+(x+xt-1)\phi_2+x=0. \tag{1.8}$$

Since the two formal power series ϕ_0 and ϕ_2 satisfy the same equation ((1.2) and (1.8) are identical) they must be equal. □

2. PROOFS OF THEOREMS 4 AND 5

Introduce the following weight on L:

$$P_0(w)=x^{n(w)}u^{h(w)}t_1^{k_1(w)}t_2^{k_2(w)},$$

and let

$$\psi_0=\psi_0(x,u,t_1,t_2):=P_0(L)\left(:=\sum_{w\in L}P_0(w)\right).$$

ψ_0 is a formal power series in the four indeterminates x,u,t_1,t_2. Of course $E_0(n,\lambda,\rho,\sigma)$ is the coefficient of $x^n u^\lambda t_1^\rho t_2^\sigma$ in $\psi_0(x,u,t_1,t_2)$. We need the following:

LEMMA 2: *ψ_0 satisfies the following quadratic equation:*

$$(x^2ut_1t_2+xut_2-x^2ut_2)\psi_0^2+(-1+x+x^2ut_1+xu-x^2u+x^2ut_1t_2-x^2ut_2)\psi_0$$
$$+(x+x^2ut_1-x^2u)=0. \tag{2.1}$$

Proof Let us *weigh* both sides of (1.1) with P_0:

$$P_0(L)=P_0(01)+P_0(01L)+P_0(0L1)+P_0(0L1L). \tag{2.2}$$

Then, reasoning as before

i) $P_0(01)=x$.

ii) $P_0(01w)=xP_0(w)$, so $P_0(01L)=x\psi_0$. (We note, for later, that $P_0(01\cup 01L)=x+x\psi_0$, and thus $P_0(L-01-01L)=\psi_0-x-x\psi_0$).

iii) If w starts with "01" then $k_1(0w1)=k_1(w)+1$ (a new "001" is born at the beginning), otherwise $k_1(0w1)=k_1(w)$. Always $n(0w1)=n(w)+1$, and $h(0w1)=h(w)+1$ and $k_2(0w1)=k_2(w)$, so:

$$P_0(0w1)=\begin{cases}xut_1P_0(w) & w\in 01\cup 01L\\ xuP_0(w) & w\in L-01-01L\end{cases}.$$

Thus

$$P_0(0L1)=xut_1(x+x\psi_0)+xu(\psi_0-x-x\psi_0).$$

iv) $k_1(0u1v)$ is equal to $k_1(u)+1+k_1(v)$ if u starts with "01", and to $k_1(u)+k_1(v)$ otherwise. In either case we have $h(0u1v)=h(u)+h(v)+1$ and $k_2(0u1v)=k_2(u)+k_2(v)+1$ (there is always a brand new "110" formed around the 1 in $0u1v$). It follows that,

$$P_0(0u1v)=\begin{cases}xut_1t_2P_0(u)P_0(v) & u\in 01\cup 01L\\ xut_2P_0(u)P_0(v) & u\in L-01-01L\end{cases}.$$

Thus,

$$P_0(0L1L)=xut_1t_2(x+x\psi_0)\psi_0+xut_2(\psi_0-x-\psi_0)\psi_0.$$

Putting it all in (2.2) yields

$$\psi_0=x+x\psi_0+xut_1(x+x\psi_0)+xu(\psi_0-x-x\psi_0)+xut_1t_2(x+x\psi_0)\psi_0$$
$$+xut_2(\psi_0-x-x\psi_0)\psi_0, \tag{2.3}$$

which yields (2.1).

Let a,b,c, be the coefficients of ψ_0^2, ψ_0, and 1 respectively in (2.1). Solving (2.1) we can get an explicit formula for ψ_0:

$$\psi_0=\frac{-b-\sqrt{\Delta}}{2a} \tag{2.4}$$

where $\Delta=b^2-4ac$ is a certain polynomial that both the diligent reader and MACSYMA will not have any trouble computing, but that is not reproduced here to save space.

THEOREM 4 (Kreweras [7], Section 2) $E_1(n,\lambda,\rho,\sigma)=E_0(n,\lambda,\rho,\sigma)$.

Proof Introduce the weight

$$P_1(w)=x^{n(w)}u^{d(w)}t_1^{h_1(w)}t_2^{h_2(w)},$$

and let

$$\psi_1(x,u,t_1,t_2):=P_1(L)\left(:=\sum_{w\in L}P_1(w)\right).$$

ψ_1 is a formal power series in 4 variables, and, of course, $E_1(n,\lambda,\rho,\sigma)$ is the coefficient of $x^nu^\lambda t_1^\rho t_2^\sigma$ in ψ_1. Theorem 4 is equivalent to the statement that the two formal power series ψ_0 and ψ_1 are identical.

Let us find an equation for ψ_1. Taking the weight P_1 on both sides of (1.1), we have:

$$P_1(L)=P_1(01)+P_1(01L)+P_1(0L1)+P_1(0L1L). \tag{2.5}$$

Now, as before P_1 $(01)=x; P_1(01L)=x\psi_1$. For any formal power series f in x,u,t_1,t_2, let

$$\hat{f}(x,u,t_1,t_2):=f(xu,1/u,t_2,t_1).$$

Obviously we have: $n(0w1)=1+n(w)$, $d(0w1)=n(w)-d(w)$, $h_1(0w1)=1+h_2(w)$, and $h_2(0w1)=h_1(w)$, so

$$P_1(0w1)=x^{n(0w1)}u^{d(0w1)}t_1^{h_1(0w1)}t_2^{h_2(0w1)}=x^{1+n(w)}u^{n(w)-d(w)}t_1^{1+h_2(w)}t_2^{h_1(w)}$$
$$=(xt_1)(xu)^{n(w)}(1/u)^{d(w)}t_2^{h_1(w)}t_1^{h_2(w)}=(xt_1)\hat{P}_1(w).$$

Thus $P_1(0L1)=xt_1\hat{\psi}_1$. Similarly, $P_1(0L1L)=xt_1\hat{\psi}_1\psi_1$. Putting it all in (2.5) yields the following functional equation for ψ_1:

$$\psi_1=x+x\psi_1+xt_1\hat{\psi}_1+xt_1\hat{\psi}_1\psi_1. \tag{2.6}$$

It is readily seen that (2.6) has a unique solution in the ring of formal power series in x, u, t_1, t_2. A purely routine calculation (that MACSYMA kindly performed for me) shows that ψ_0, as given by (2.4), also satisfies (2.6). By uniqueness it follows that the two formal power series ψ_0 and ψ_1 are identical.

THEOREM 5 (Kreweras [7], Section 3) $E_2(n,\lambda,\rho,\sigma)=E_0(n,\lambda,\rho,\sigma)$.

Proof Introduce the weight

$$P_2(w)=x^{n(w)}u^{d(w)}t_1^{k_1(w)}t_2^{k_2(w)},$$

and let

$$\psi_2(x,u,t_1,t_2):=P_2(L)\left(:=\sum_{w\in L}P_2(w)\right).$$

ψ_2 is a formal power series in 4 variables, and, of course, $E_2(n,\lambda,\rho,\sigma)$ is the coefficient of $x^n u^\lambda t_1^\rho t_2^\sigma$ in ψ_2. Theorem 5 is equivalent to the statement that the two formal power series ψ_0 and ψ_2 are identical.

Let us find an equation for ψ_2. Taking the weight P_2 on both sides of (1.1), we have:

$$P_2(L)=P_2(01)+P_2(01L)+P_2(0L1)+P_2(0L1L). \tag{2.7}$$

Now, as before P_2 $(01)=x; P_2(01L)=x\psi_2$. For any formal power series f in x,u,t_1,t_2, let

$$\bar{f}(x,u,t_1,t_2):=f(xu,1/u,t_1,t_2).$$

Obviously we have:

$$n(0w1)=1+n(w), d(0w1)=n(w)-d(w), k_1(0w1)=1+k_1(w),$$

if w starts with "01", otherwise $k_1(0w1)=k_1(w)$. In either case we have $k_2(0w1)=k_2(w)$, so

$$P_2(0w1)=\begin{cases} xt_1\overline{P_2(w)} & w\in 01\cup 01L \\ x\overline{P_2(w)} & w\in L-01-01L \end{cases}.$$

Thus,

$$P_2(0L1)=(xt_1)(xu+xu\bar{\psi}_2)+x(\bar{\psi}_2-xu-xu\bar{\psi}_2).$$

Similarly,

$$P_2(0u1v)=\begin{cases} xt_1t_2\overline{P_2(u)}P_2(v) & u\in 01\cup 01L \\ xt_2\overline{P_2(u)}P_2(v) & u\in L-01-01L \end{cases}.$$

Thus,

$$P_2(0L1L)=(xt_1t_2)(xu+xu\bar{\psi}_2)\psi_2+xt_2(\bar{\psi}_2-xu-xu\bar{\psi}_2)\psi_2.$$

Putting it all together in (2.7) yields:

$$\begin{aligned}\psi_2=x+x\psi_2+xt_1(xu+xu\bar{\psi}_2)+x(\bar{\psi}_2-xu-xu\bar{\psi}_2)\\ +xt_1t_2(xu+xu\bar{\psi}_2)\psi_2+xt_2(\bar{\psi}_2-xu-xu\bar{\psi}_2)\psi_2.\end{aligned} \tag{2.8}$$

A purely routine calculation (that I performed on MACSYMA) shows that ψ_0, as given by (2.4), satisfies the functional equation (2.8). By uniqueness it follows that $\psi_2=\psi_0$. □

3. PROOFS OF THEOREMS 3 AND 6

For any Laurent formal power series $f(u_1,\ldots,u_n)$, let C.T.f (pronounced "Constant Term of f") be the coefficient of $u_1^0\ldots u_n^0$ in f. Let us recall the celebrated *Lagrange inversion formula*:

THE LAGRANGE INVERSION FORMULA (e.g. [4], p. 17)

Let $f(X)$ be any formal power series with a non-zero constant term, and let $Y=X/f(X)$, then there exists a formal power series g such that $X=g(Y)$ and the coefficient of Y^n in $X=g(Y)$ is given by

$$C.T.\frac{X}{Y^n}=\frac{1}{n}C.T.\frac{(f(X))^n}{X^{n-1}}.$$

THEOREM 3 (Narayana [10])

$$N_0(n,\alpha)=\frac{1}{n}\binom{n}{\alpha}\binom{n}{\alpha+1}.$$

Proof Rewrite (1.2) as

$$x=\frac{\phi_0}{(t\phi_0+1)(\phi_0+1)}.$$

Treat $\phi_0(x,t)$ as a formal power series in x and demote t to the status of a parameter. By the Lagrange inversion formula:

$$N_0(n,\alpha)=C.T._{x,t}\frac{\phi_0}{x^n t^\alpha}=\frac{1}{n}C.T._{\phi_0,t}\frac{[(t\phi_0+1)(\phi_0+1)]^n}{\phi_0^{n-1}t^\alpha}$$

$$=\frac{1}{n}\binom{n}{\alpha}C.T._{\phi_0}\frac{\phi_0^\alpha(\phi_0+1)^n}{\phi_0^{n-1}}=\frac{1}{n}\binom{n}{\alpha}C.T._{\phi_0}\frac{(\phi_0+1)^n}{\phi_0^{n-\alpha-1}}=\frac{1}{n}\binom{n}{\alpha}\binom{n}{\alpha+1}.$$

THEOREM 6 (Kreweras–Poupard [7])

$$E_0(n,\lambda,\rho,\sigma)=\frac{1}{\lambda}\binom{\lambda}{\rho}\binom{\lambda}{\sigma}\binom{n-\lambda-1}{\rho-1}\binom{n-\lambda}{\sigma+1}.$$

Proof It is readily seen that (2.1) can be rewritten as

$$u=\frac{\psi_0}{\psi_0(1+\psi_0 t_2)x[(\psi_0+1)t_1x+(\psi_0-x\psi_0-x)]/(\psi_0-x\psi_0-x)}.$$

Let us view ψ_0 as a formal power series in u and consider x, t_1, and t_2 as parameters. Lagrange inversion yields:

$$E_0(n,\lambda,\rho,\sigma)=C.T._{x,u,t_1,t_2}\frac{\psi_0(x,u,t_1,t_2)}{x^n u^\lambda t_1^\rho t_2^\sigma}$$

$$=\frac{1}{\lambda}C.T._{\psi_0,x,t_1,t_2}\left[\frac{\psi_0^\lambda(1+\psi_0t_2)^\lambda x^\lambda[(\psi_0+1)t_1x+(\psi_0-x\psi_0-x)]^\lambda}{\psi_0^{\lambda-1}t_2^\sigma x^n t_1^\rho(\psi_0-\psi_0x-x)^\lambda}\right]$$

$$=\frac{1}{\lambda}C.T._{\psi_0,x}\frac{\psi_0\binom{\lambda}{\sigma}\psi_0^\sigma x^{\lambda-n}\binom{\lambda}{\rho}(\psi_0+1)^\rho x^\rho(\psi_0-x\psi_0-x)^{\lambda-\rho}}{(\psi_0-x\psi_0-x)^\lambda}$$

$$=\frac{1}{\lambda}\binom{\lambda}{\sigma}\binom{\lambda}{\rho}C.T._{\psi_0,x}\left[\frac{\psi_0^{\sigma+1}(\psi_0+1)^\rho[\psi_0-x(\psi_0+1)]^{-\rho}}{x^{n-\lambda-\rho}}\right]$$

$$=\frac{1}{\lambda}\binom{\lambda}{\sigma}\binom{\lambda}{\rho}C.T._{\psi_0,x}\left[\frac{\psi_0^{\sigma+1}(\psi_0+1)^\rho\psi_0^{-\rho}[1-x(\psi_0+1)/\psi_0]^{-\rho}}{x^{n-\lambda-\rho}}\right]$$

$$=\frac{1}{\lambda}\binom{\lambda}{\sigma}\binom{\lambda}{\rho}C.T._{\psi_0}\left[\psi_0^{\sigma-\rho+1}(\psi_0+1)^\rho\binom{-\rho}{n-\lambda-\rho}(-1)^{n-\lambda-\rho}\right.$$

$$\times [(\psi_0+1)/\psi_0]^{n-\lambda-\rho}\Bigg]$$

$$=\frac{1}{\lambda}\binom{\lambda}{\sigma}\binom{\lambda}{\rho}\binom{n-\lambda-1}{n-\lambda-\rho}C.T._{\psi_0}\frac{(\psi_0+1)^{n-\lambda}}{\psi_0^{n-\lambda-\sigma-1}}$$

$$=\frac{1}{\lambda}\binom{\lambda}{\sigma}\binom{\lambda}{\rho}\binom{n-\lambda-1}{\rho-1}\binom{n-\lambda}{\sigma+1}.$$

4. ORDERED TREES

There is a well known bijection, Φ, between ordered trees with $n+1$ vertices and legal bracketings of length $2n$ (e.g. [14], p. 61–62, [3]). It is given recursively by $\Phi(\cdot)=$"empty word", and $\Phi(r; T_1,\ldots,T_\alpha)=0\Phi(T_1)1\ldots 0\Phi(T_\alpha)1$. Alternatively, $\Phi(T)$ can be obtained by transversing T in *preorder* ([6], p. 334), writing a "0" whenever we go down, and a "1" whenever we backtrack up. An *internal vertex* is a vertex that is not a leaf. A vertex is called *leftmost* it it is the leftmost vertex among its brothers. A vertex is called *rightmost* if it is the rightmost vertex among its brothers. The *height* of a vertex is the length of the path that connects it to the root. It is readily seen that the parameters h_1, h_2, d, k_1, k_2 translate as follows:

$h_1(T)=$ # of internal vertices of odd height of T.

$h_2(T)=$ # of internal vertices of even height of T. (Not counting the root).

$h(T)=$ # of internal vertices of T (not counting the root).

$d(T)=$ # of internal vertices of T of even height (not counting the root).

$k_1(T)=$ # of leaves in T that are leftmosts.

$k_2(T)=$ # of internal vertices in T that are non-rightmosts.

Theorems 1–6 thus translate into statements about tree enumeration. I will now state two corollaries that I believe to be of some interest. The first one is obtained by a routine Vandermonde–Chu summation ([6], p. 58) of E_0, with respect to σ, on the identity of Theorem 6. The second is obtained by a further Vandermonde–Chu summation of ρE_0 with respect to ρ, and dividing by $N_0(n,\lambda)$.

COROLLARY 1 *The number of ordered trees with $n+1$ vertices, λ leaves, and ρ leaves that are leftmosts, is*

$$\frac{1}{\lambda}\binom{\lambda}{\rho}\binom{n-\lambda-1}{\rho-1}\binom{n}{\lambda+1}.$$

COROLLARY 2 *Among all ordered trees with $n+1$ vertices and λ leaves, the average number of leaves that are leftmosts is $\lambda(n-\lambda)/(n-1)$.*

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Corollary 2 has the following "computer science" application. In the UNIX operating system, suppose that in your HOME directory you have λ files and $n-\lambda$ directories ("file folders"). Then the expected number of files that have the property that they appear on the top of the list when you do "1*s*" on their immediate directory is $\lambda(n-\lambda)/(n-1)$.

5. ENCORE

According to the taxonomy of *Mathematical Reviews*, combinatorics is divided into three branches: "Enumerative Combinatorics", "Designs", and "Graph Theory" (05A, 05B, and 05C respectively). The founding fathers of these branches were, respectively, Leibnitz, Kirkman, and Euler. While Leibnitz and Euler were masters of all trades, Kirkman, in the 19th century, was already a "narrow minded specialist" who specialized in combinatorics. However specialization then did not yet reach its modern extremes, and the creator of 05B also dabbled in 05A. In 1857, Kirkman [5] considered the problem of enumerating the divisions of a convex polygon by non-crossing diagonals. He stated the formula

$$\frac{1}{m+1}\binom{k-3}{m}\binom{k+m-1}{m} \tag{5.1}$$

for the number, call it $D(k,m)$, of distinct ways of adding m non-crossing diagonals to a convex polygon of k sides taken in fixed order. Of course when $m=k-3$ this gives the well known fact (e.g. [11], Section 2.1, [12] pp. 25–30) that the number of ways of triangularizing a k-sided convex polygon by non-intersecting diagonals is Catalan's number $(2k-4)!/(k-1)!(k-2)!$.

Kirkman's formula was only proved forty years later, by Cayley [1] (see also Watson [19]). Here I give a short proof that perfectly fits the spirit of this paper. I learned about this formula while I was browsing through Temperley [15]. I urge the readers to look up (or better still, buy) this fascinating book that, among many other gems, contains a beautiful discussion of how this seemingly naive problem is closely related to the properties of liquids!

Let P be the set of all configurations consisting of a convex polygon (in which one side is distinguished by, say, being the "bottom horizontal" side) with some (possibly zero) non-intersecting diagonals. For any such configuration p, let

$k(p) := $ number of sides of p,

$m(p) := $ number of diagonals of p, and

$\mathit{weight}\ (p) = z^{k(p)} w^{m(p)}$,

$$\phi = \text{weight } (P)\left(:= \sum_{p \in P} \mathit{weight}(p)\right).$$

For any such configuration, deleting the bottom (distinguished) edge will result in a certain number, $l \geq 2$, arranged clockwise of entities each of which is either a single edges (weight z), or a smaller configuration. In the later case the act of deleting the bottom edge of the original configuration turned one of the diagonals into an edge. Thus the total weight due to a single component is $z+w\phi/z$. Summing over all conceivable $l\geqq 2$, not forgetting to give the deleted bottom edge its due credit, gives:

$$\phi = z\sum_{l=2}^{\infty}(z+w\phi/z)^l. \tag{5.2}$$

It is convenient to introduce

$$\psi = z + w\phi/z \tag{5.3}$$

(5.2) gives the following quadratic equation for ψ

$$(1+w)\psi^2-(1+z)\psi+z=0. \tag{5.4}$$

Solving (5.4) and substituting in (5.3) yields Watson's [19] formula for ϕ that is reproduced in Temperley's [15] beautiful book. (5.4) is equivalent to

$$w=\frac{\psi}{z^3/(1-\psi)(\psi-z)} \tag{5.5}$$

Lagrange inverting (5.5) with respect to w gives:

$$D(k,m)=C.T._{z,w}\frac{\phi}{z^k w^m}=C.T._{z,w}\frac{\psi}{z^{k-1}w^{m+1}}=\frac{1}{m+1}C.T._{z,\psi}\frac{[\psi^3/(1-\psi)(\psi-z)]^{m+1}}{z^{k-1}\psi^m}$$

$$=\frac{1}{m+1}C.T._{z,\psi}\frac{\psi^{2m+3}(1-\psi)^{-m-1}\psi^{-m-1}(1-z/\psi)^{-m-1}}{z^{k-1}}$$

$$=\frac{1}{m+1}C.T._{\psi}\psi^{m+2}(1-\psi)^{-m-1}(-1/\psi)^{k-1}\binom{-m-1}{k-1}$$

$$=\frac{1}{m+1}\binom{m+k-1}{k-1}C.T._{\psi}\frac{(1-\psi)^{-m-1}}{\psi^{k-m-3}}$$

$$=\frac{1}{m+1}\binom{m+k-1}{k-1}\binom{k-3}{k-m-3}=\frac{1}{m+1}\binom{k-3}{m}\binom{k+m-1}{m}.$$

According to Watson [19], neither Kirkman [5] nor Cayley [1] were able to prove a certain "summatory formula" that he then proceeded to prove. This formula is equivalent, in terms of the generating function ϕ to

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$$2\frac{\partial \phi}{\partial w}=z\frac{\partial}{\partial z}\left(\frac{\phi}{z}\right)^2. \tag{5.7}$$

This is immediate both on combinatorial grounds and by directly substituting into (5.7) the explicit expression for phi obtained by solving (5.4) and substituting in (5.3).

As a closing remark let us note that $D(k,m)$ also counts the number of ordered trees with k leaves and $m+1$ internal vertices in which no vertex is allowed to have exactly one son.

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