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A COMBINATORIAL PROOF OF DYSON'S CONJECTURE

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Dyson's conjecture, already proved by Gunson, Wilson and Good, is given a direct combinatorial proof.

1. Introduction

We are going to give a combinatorial proof of

Dyson's conjecture [2]. *The constant term of* $\prod_{1\le i\ne j\le n}(1-x_i/x_j)^{a_i}$ *is*

$$(a_1+\cdots+a_n)!/a_1!\cdots a_n!.$$

This conjecture was proved by Gunson [6], Wilson [10] and Good [5]. Recently Andrews [1] and MacDonald [7] formulated certain generalizations of Dyson's conjecture which are still open. Andrews conjectured a q-analog, and MacDonald raised a conjecture on root systems of simple Lie algebras.

A related conjecture was raised by Mehta [8] and turned out to follow from a result of Selberg [9] which lay dormant for several decades. MacDonald [7] also formulated a generalization of Mehta's conjecture for Lie algebras. Regev and Beckner used Selberg's result to prove this conjecture for all the classical Lie algebras, and MacDonald did the same for Dyson's conjecture.

None of the proofs of Dyson's conjecture, of which Good's [5] is particularly elegant, seem to generalize. We believe that our proof has a good chance of being generalized, because most combinatorial proofs involving binomial coefficients have q-analogs. However, another idea is still needed since the obvious q-generalization of our proof fails.

We will prove Dyson's conjecture in the following equivalent form:

Theorem. *Let* $G(a_1,\ldots,a_n)$ *be the coefficient of* $x_1^{(n-1)a_1}x_2^{(n-1)a_2}\cdots x_n^{(n-1)a_n}$ *in* $\prod_{1\le i<j\le n}(x_i-x_j)^{a_i+a_j}$, *then*

$$G(a_1,\ldots,a_n)=(-1)^{a_2+2a_3+\cdots+(n-1)a_n}(a_1+\cdots+a_n)!/a!\cdots a_n!.$$

The case $a_1=\cdots=a_n=1$ follows from the Vandermonde determinant identity

$\det(x_i^{j-1}) = \prod_{1\le i<j\le n}(x_i - x_j)$ and the case $n=3$ is equivalent to Dixon's identity [1 p. 214]. Gessel [4] gave a combinatorial proof of Vandermonde's determinant identity and Foata [3] gave a combinatorial proof of Dixon's identity. The present proof is a joint generalization of these proofs and freely draws from their ideas.

Before embarking on the proof we must admit that our proof uses Vandermonde's determinant identity and therefore is purely combinatorial only when taken together with Gessel's proof.

2. Proof of Dyson's conjecture

A *tournament* is a strictly upper triangular matrix $T=(t_{ij})_{1\le i<j\le n}$ where each t_{ij} is either i or j. If $t_{ij}=i$ (respectively j) we say that "i beat j" (respectively "j beat i"). If we assume that before the tournament started "1" was the best, "2" the second, ..., "n" the worst, then $t_{ij}=j$ is certainly a *surprise*. Let $\sigma(T)$ be the number of surprises of T. A tournament is *transitive* if it induces a clear cut ranking of the players; namely, there is a player $\pi(1)$ who beat all the rest, a player $\pi(2)$ who beat everybody except $\pi(1)$, ..., and a player $\pi(n)$ who got beaten by all. Since there are $\binom{n}{2}$ games each of which has two outcomes, there are $2^{\binom{n}{2}}$ possible tournaments, $n!$ of which are transitive.

The weight of a tournament $T=(t_{ij})$ is defined as

$$w(T) = (-1)^{\sigma(T)} \prod_{1\le i<j\le n} x_{t_{ij}}$$

(this is a monomial $(-1)^{\sigma(T)}x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ where α_i is the number of games won by "i"). The weight of a transitive tournament corresponding to a permutation π is

$$(-1)^{\sigma(\pi)}x_{\pi(1)}^{n-1}x_{\pi(2)}^{n-2}\cdots x_{\pi(n-1)}^{1},$$

where $\sigma(\pi)$ is the number of inversions of π ($\pi(1)$ won $n-1$ games, $\pi(2)$ won $n-2$ games, etc.). Now expanding $\prod_{1\le i<j\le n}(x_i-x_j)$ lists the weights of all the possible $2^{\binom{n}{2}}$ tournaments (each of the $\binom{n}{2}$ terms of the product corresponds to a game, if "i" won we take x_i, if "j" won we take $-x_j$). By the famous Vandermonde determinant identity

$$\prod_{1\le i<j\le n}(x_i-x_j) = \det(x_i^{j-1}) = \sum (-1)^{\sigma(\pi)}x_{\pi(1)}^{n-1}\cdots x_{\pi(n-1)}^{1}. \tag{1}$$

The right hand side of (1) lists the weights of all the transitive tournaments so it turns out that the weights of the non-transitive tournaments cancel each other. This implies

Proposition 1. *There is an involution α from the non-transitive tournaments onto themselves such that $w(\alpha(T)) = -w(T)$.*

What is listed by $\prod_{1\le i<j\le n}(x_i-x_j)^{a_i+a_j}$? Now every pair of players had a_i+a_j games and this motivates the definition of a *multi-tournament* of type $(a_1,\ldots,a_n)$ as a strictly upper triangular matrix $M=(m_{ij})_{1\le i<j\le n}$ such that m_{ij} is a word of length a_i+a_j in $\{i,j\}$. Let $S(a_1,\ldots,a_n)$ be the set of such multi-tournaments of weight $\pm x_1^{(n-1)a_1}\cdots x_n^{(n-1)a_n}$, i.e. $M=(m_{ij})$ such that the number of occurrences of "i" in all the m_{ij}'s is $(n-1)a_i$, $(i=1,\ldots,n)$. As before let $\sigma(M)$ be the number of surprises (occurrences of "j" in m_{ij}) then it is readily seen that $G(a_1,\ldots,a_n)$ of the theorem is given by

$$G(a_1,\ldots,a_n)=\sum_{M\in S(a_1,\ldots,a_n)}(-1)^{\sigma(M)}. \tag{2}$$

Let $C(a_1,\ldots,a_n)$ be the set of words in $\{1,\ldots,n\}$ with a_1 1's, $\ldots$, a_n n's. It is well known and easy to see that $C(a_1,\ldots,a_n)$ has $(a_1+\cdots+a_n)!/a_1!\cdots a_n!$ elements. Define a mapping $\varphi: C(a_1,\ldots,a_n)\to S(a_1,\ldots,a_n)$ by $(\varphi(v))_{ij}=$ word obtained from v by ignoring everything but the letters i and j. For example

$$\varphi(412433)=\begin{pmatrix} m_{12}=12, & m_{13}=133, & m_{14}=414 \\ & m_{23}=233, & m_{24}=424 \\ & & m_{34}=4433 \end{pmatrix}.$$

Let $S_g(a_1,\ldots,a_n)=\varphi(C(a_1,\ldots,a_n))$ be called the set of good guys and its complement in $S(a_1,\ldots,a_n)$ the set of bad guys. Note that a good guy $M=\varphi(v)$ has the property that the matrix of the heads of M is a transitive tournament, namely the one corresponding to the permutation obtained by taking the n first appearances of each of the letters of our alphabet $\{1,\ldots,n\}$ (in the above example we get the permutation 4123 corresponding to the head tournament

$$\begin{pmatrix} 1 & 1 & 4 \\ & 2 & 4 \\ & & 4 \end{pmatrix}).$$

Given a good guy M it is possible to find its origin, namely the word v such that $\varphi(v)=M$. The first letter of v *must* appear in $n-1$ places in the tournament of the heads of m_{ij}, hence must be its winner. Having found it we cross the $n-1$ occurrences of this letter in the heads and look at the remaining multi-tournament the winner of whose current heads tournament must be the second letter of v. We repeat the process of crossing out and looking for the winners, until we completely decipher v. If at some stage the tournament of heads is non-transitive it means that our multi-tournament is bad.

Example. The winner of the current tournament of heads has been underlined.

$$v=\cdots,\qquad M=\begin{pmatrix} 12 & 133 & \underline{4}14 \\ & 233 & \underline{4}24 \\ & & \underline{4}433 \end{pmatrix},$$

$$v=4\cdots,\qquad M=\begin{pmatrix} \underline{1}2 & \underline{1}33 & \underline{1}4 \\ & 233 & 24 \\ & & 433 \end{pmatrix},$$

$$v=41\cdots,\qquad M=\begin{pmatrix} \underline{2} & 33 & 4 \\ & \underline{2}33 & \underline{2}4 \\ & & 433 \end{pmatrix},$$

$$v=412,\qquad M=\begin{pmatrix} \text{empty} & 33 & \underline{4} \\ & 33 & \underline{4} \\ & & \underline{4}33 \end{pmatrix},$$

$$v=4124,\qquad M=\begin{pmatrix} \text{empty} & \underline{3}3 & \text{empty} \\ & \underline{3}3 & \text{empty} \\ & & \underline{3}3 \end{pmatrix},$$

$$v=41243,\qquad M=\begin{pmatrix} \text{empty} & \underline{3} & \text{empty} \\ & \underline{3} & \text{empty} \\ & & \underline{3} \end{pmatrix},$$

$$v=412433.$$

It follows that φ is one-one and that there are $(a_1+\cdots+a_n)!/a_1!\cdots a_n!$ good guys. We now claim that all the good guys have the same number of surprises: $a_2+2a_3+\cdots+(n-1)a_n$. Indeed every letter "i" is placed in $n-1$ places of which $i-1$ cause surprises. Since there are a_i occurrences of "i" it contributes $(i-1)a_i$ surprises $(i=2,\ldots,n)$.

Now by (2)

$$G(a_1,\ldots,a_n)=\sum_{M\text{ good}}(-1)^{\sigma(M)}+\sum_{M\text{ bad}}(-1)^{\sigma(M)}. \tag{3}$$

But $(-1)^{\sigma(M)}=(-1)^{a_2+\cdots+(n-1)a_n}$ for all the $(a_1+\cdots+a_n)!/a_1!\cdots a_n!$ good guys; thus

$$G(a_1,\ldots,a_n)=(-1)^{a_2+2a_3+\cdots+(n-1)a_n}(a_1+\cdots+a_n)!/a_1!\cdots a_n! + \sum_{M\text{ bad}}(-1)^{\sigma(M)}.$$

It remains to show that

$$\sum_{M\text{ bad}}(-1)^{\sigma(M)}=0. \tag{4}$$

For this purpose we will define an involution ψ from the bad guys onto themselves such that $(-1)^{\sigma(\psi(M))}=-(-1)^{\sigma(M)}$. Given a bad M we apply the above process for finding the origin of a good guy until we encounter a non-transitive head tournament. We replace that non-transitive tournament by its image under α of

Proposition 1. Since α preserves weight, $\psi(M)$ is still in $S(a_1,\ldots,a_n)$ and $(-1)^{\sigma(\psi(M))}=-(-1)^{\sigma(M)}$. Also it is clear that $\psi(M)$ is a bad guy and $\psi^2=\text{identity}$ since $\alpha^2=\text{identity}$. This completes the proof since there are as many $+1$'s as -1's in (4).

Example. If $n=3$ the only non-transitive tournaments are $\begin{pmatrix}1 & 3\\ & 2\end{pmatrix}$ and $\begin{pmatrix}2 & 1\\ & 3\end{pmatrix}$ the first having one surprise and the second two surprises. α is given by

$$\alpha\begin{pmatrix}1 & 3\\ & 2\end{pmatrix}=\begin{pmatrix}2 & 1\\ & 3\end{pmatrix},\qquad \alpha\begin{pmatrix}2 & 1\\ & 3\end{pmatrix}=\begin{pmatrix}1 & 3\\ & 2\end{pmatrix}.$$

Now consider

$$M=\begin{pmatrix}12112 & 13311\\ & 2223\end{pmatrix};$$

applying the above process yields

$$\begin{pmatrix}\underline{1}2112 & \underline{1}3311\\ & 2223\end{pmatrix}\to\begin{pmatrix}\underline{2}112 & 3311\\ & \underline{2}223\end{pmatrix}\to\begin{pmatrix}112 & 3311\\ & 223\end{pmatrix};$$

at this stage the head tournament, $\begin{pmatrix}1 & 3\\ & 2\end{pmatrix}$, is non-transitive and we replace it by $\alpha\begin{pmatrix}1 & 3\\ & 2\end{pmatrix}=\begin{pmatrix}2 & 1\\ & 3\end{pmatrix}$ getting

$$\begin{pmatrix}212 & 1311\\ & 323\end{pmatrix};$$

now put back the previously erased letters, getting

$$\psi(M)=\begin{pmatrix}12212 & 11311\\ & 2323\end{pmatrix}.$$

References

[1] G.E. Andrews, Problems and prospects for basic hypergeometric functions, in: R.A. Askey, ed., Theory and Applications of Special Functions (Academic Press, New York, 1975) 191–224.

[2] F.J. Dyson, Statistical theory of the energy levels of complex systems I, J. Math. Phys. 3 (1962) 140–156.

[3] P. Cartier and D. Foata, Problémes combinatoires de commutation et réarrangements, Lecture Notes in Mathematics 85 (Springer-Verlag, Berlin, 1969).

[4] I.M. Gessel, Tournaments and Vandermonde's determinants, Preprint, M.I.T. (1979).

[5] I.J. Good, Short proof of a conjecture of Dyson, J. Math. Phys. 11 (1970) 1884.

[6] J. Gunson, Proof of a conjecture of Dyson in the statistical theory of energy levels, J. Math. Phys. 3 (1962) 752–753.

[7] I.G. MacDonald, Some conjectures for root systems and finite reflection groups, Preprint, Queen Mary College (1980).

[8] M.L. Mehta, Random matrices and the Statistical Theory of Energy Levels (Academic Press, New York, 1967).

[9] A. Selberg, Bemerkninger om et multiplet integral, Noisk Mat. Tidsskr. 26 (1944) 71–78.

[10] K. Wilson, Proof of a conjecture by Dyson, J. Math. Phys. 3 (1962) 1040–1043.