J. Symbolic Computation (1991) **11**, 195–204

The Method of Creative Telescoping

DORON ZEILBERGER

Department of Mathematics and Computer Science, Temple University, Philadelphia, PA 19122, *USA*

In memory of John Riordan, master of ars combinatorica

(*Received* 1 *June* 1989)

An algorithm for definite hypergeometric summation is given. It is based, in a non-obvious way, on Gosper's algorithm for definite hypergeometric summation, and its theoretical justification relies on Bernstein's theory of holonomic systems.

1. Introduction

In Zeilberger (preprint) it was shown that Joseph N. Bernstein's theory of holonomic systems (Bernstein, 1971; Bjork, 1979) forms a natural framework for proving a very large class of special function identities. A very general, albeit slow, algorithm for proving any such identity was given. In Zeilberger (to appear) a much faster algorithm was given for the important special case of hypergeometric sums

$$\sum_k F(n, k),$$

where the summand $F(n, k)$ is hypergeometric in both n and k, (i.e. both $F(n+1, k)/F(n, k)$ and $F(n, k+1)/F(n, k)$ are rational functions of n and k.)

In the present paper this fast algorithm is described in much more detail and several applications to combinatorics, probability, computation, logic and orthogonal polynomials are given. A listing of a MAPLE program implementing the algorithm is available from the author, and will appear elsewhere (Zeilberger, submitted).

Suppose we are given a certain discrete function of two variables $F(n, k)$, and it is required to prove that the sequence $a(n)$, defined by

$$a(n) := \sum_k F(n, k),$$

satisfies a certain homogeneous linear recurrence equation:

$$s_0(n)a(n) + s_1(n)a(n+1) + \cdots + s_L(n)a(n+L) = 0. \tag{1}$$

The method of *creative telescoping* proceeds by "cleverly constructing" another discrete

Supported in part by NSF grant DMS8800663.

0747-7171/91/030195+10 $03.00/0

function $G(n, k)$ that satisfies

$$s_0(n)F(n,k)+s_1(n)F(n+1,k)+\cdots+s_L(n)F(n+L,k)=G(n,k)-G(n,k+1), \quad (2)$$

and then (1) follows upon summing (2) w.r.t. to k.

The term "creative telescoping" was coined, as far as I know, by A. van der Poorten (1979) in his delightful account of Apery's proof of the irrationality of $\zeta(3)$. There it was required to show that

$$a(n):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

satisfies the recurrence

$$(n+1)^3a(n+1)-(34n^3+51n^2+27n+5)a(n)+n^3a(n-1)=0,$$

where,

$$G(n,k):=4(2n+1)((k(2k+1)-(2n+1)^2)\binom{n}{k}^2\binom{n+k}{k}^2.$$

For a general $F(n, k)$ there is no reason to expect that $a(n):=\sum_k F(n,k)$, should satisfy any linear recurrence equation with "nice" coefficients. Furthermore when it does, there is no guarantee that such a miraculous $G(n, k)$ that satisfies (2) exists. However, if $F(n, k)$ is *holonomic* (see below), then, as was proved in Zeilberger (preprint), $a(n)$ always satisfies a linear recurrence equation of the form (1), where the coefficients s_i are *polynomials* in n. The general "slow" algorithm of Zeilberger yields the recurrence, and in principle it also enables one to find $G(n, k)$. However, in the general holonomic case, since $F(n, k)$ is not "nice" (i.e. not hypergeometric), $G(n, k)$ is not nice either, and it is very cumbersome to present $G(n, k)$. On the other hand, if $F(n, k)$ is "nice" so is $G(n, k)$! We will show later that if $F(n, k)$ is hypergeometric, so is $G(n, k)$, and the fast algorithm that we will describe inputs $F(n, k)$ and outputs the recurrence and its *proof* at the same time. The proof consist of presenting the "certificate" $G(n, k)$ by which the readers (or their computers) proceed to verify the purely routine identity (2).

2. The Theoretical Foundation

It was proved in Zeilberger (preprint) that whenever $F(n, k)$ is *holonomic*, (in particular if both $F(n, -)$ and $F(-, k)$ satisfy linear recurrences with polynomial coefficients that are "independent" in a certain technical sense) we are *guaranteed* that $a(n)$ is holonomic in n, i.e. satisfies a homogeneous linear recurrence equation of the form (1), *with polynomial coefficients*. Let us now recall this algorithm.

Let N and K be the shift operators in n and k respectively: $N(A(n,k)):=A(n+1,k)$, $K(A(n,k)):=A(n,k+1)$, for any discrete function $A(n, k)$. Of course $N^iK^jA(n,k)=A(n+i,k+j)$. Suppose you know that $F(n, k)$ is annihilated by two linear partial difference operators $P(N,K,n,k)$ and $Q(N,K,n,k)$ with polynomial coefficients. ($F(n, k)$ is *annihilated* by an operator $R(N,K,n,k)$ if $R(N,K,n,k)F(n,k)\equiv 0$, for example the Fibonacci sequence is annihilated by the operator N^2-N-I.) It was shown in Zeilberger (preprint), following Bernstein (1971), that if $\{P, Q\}$ generate a holonomic system (i.e. are "independent") there exist operators $P_1(N,K,n,k)$ and $Q_1(N,K,n,k)$ and $R(N,K,n,k)$ such that

$$\begin{aligned}S(N,n):=&P_1(N,K,n,k)P(N,K,n,k)\\&+Q_1(N,K,n,k)Q(N,K,n,k)+(K-1)R(N,K,n,k)\end{aligned} \quad (3)$$

does not involve k and K. The algorithm given in Zeilberger did in fact produce an $R(N, K, n, k)$ that did not involve k, and thus gave us more than we needed, which partly explains why it was so slow. Now let

$$G(n, k) := R(N, K, n, k)F(n, k). \tag{4}$$

By applying (3) on $F(n, k)$ we get

$$S(N, n)F(n, k) = (K-1)G(n, k). \tag{5}$$

It follows that

$$a(n) := \sum_k F(n, k)$$

satisfies the linear recurrence equation with polynomial coefficients $S(N, n)a(n) \equiv 0$. Indeed

$$\begin{aligned} S(N, n)a(n) &= S(N, n)\sum_k F(n, k) = \sum_k S(N, n)F(n, k) \\ &= \sum_k (K-1)G(n, k) = \sum_k [G(n, k+1) - G(n, k)] = 0. \end{aligned}$$

3. Gosper's Missed Opportunity

In his celebrated essay, Freeman Dyson (1972) describes several historical missed opportunities that were caused by the reluctance of mathematicians and physicists to communicate. As an extreme case of a missed opportunity, from "his own trivial experience", Dyson narrates how he missed discovering the Macdonald-Weyl identities because "Dyson the amateur number theorist failed to talk to Dyson the physicist". (As it turned out, the identities that Dyson discovered, generalizing Jacobi's triple product identity to more products, had their number of products equal to dimensions of simple Lie algebras. Dyson as a physicist was very familiar with Lie algebras, but he missed seeing the connection.)

Something similar happened to Bill Gosper. Gosper was interested both in *indefinite summation* and *definite summation*, but apparently Gosper the "definite summer" failed to talk to Gosper the "indefinite summer", or else he would have discovered how to extend his algorithm for *indefinite hypergeometric summation* to that of *definite hypergeometric summation*. If the two Gospers would have talked to each other they (or rather he) would have reasoned as follows.

In Gosper (1978) (see also Lafron (1983) and Graham *et al.* (1989)), Gosper gives a decision procedure that given a hypergeometric sequence $a(n)$ (meaning that $a(n+1)/a(n)$ is a rational function), decides whether $\sum_{i=0}^{n} a(i)$ is also hypergeometric (up to an additive constant.) In other words Gosper's algorithm decides whether there exists a hypergeometric sequence $A(n)$ such that

$$a(n) = A(n+1) - A(n),$$

and actually produces the $A(n)$ in the affirmative case.

On the other hand Gosper the *definite summer* had developed amazing heuristic techniques for discovering "strange" hypergeometric identities of the form

$$\sum_k F(n, k) = R(n), \tag{6}$$

where both $F(n, k)$ and $R(n)$ are of "closed form" (i.e. hypergeometric). For the most part he was unable to give rigorous proofs and labelled them by "TFAPP" (True For All Practical Purposes). Most of these identities were subsequently proved by Gessel & Stanton (1982) by a variety of interesting techniques, most notably Lagrange inversion.

Superficially, nice *definite* sums have nothing whatsoever to do with nice *indefinite* sums. Indeed for most definite sums (6), for which $R(n)$ is "nice", the corresponding indefinite sum

$$R(n, k) := \sum_{i=0}^{k} F(n, i) \tag{7}$$

does not evaluate in closed form, and therefore if we try to apply Gosper's algorithm to $F(n, k)$, w.r.t. the variable k, we would get that there is no closed form anit-difference. In fact, whenever (7) is indefinitely summable one may consider the implied definite identity (6) as "trivial".

However Gosper could have proceeded as follows. For any conjectured identity (6), with $R(n)$ of closed form let

$$\frac{R(n+1)}{R(n)} = \frac{p(n)}{q(n)},$$

where we know that $p(n)$ and $q(n)$ are polynomials. Let $L(n)$ be the left side of (6). We have to prove that $L(n) = R(n)$, and since it is easy to check that $L(0) = R(0)$ we have to prove

$$p(n)L(n) - q(n)L(n+1) = 0,$$

or equivalently that

$$\sum_k [p(n)F(n, k) - q(n)F(n+1, k)] = 0.$$

It is easy to see that if $F(n, k)$ is hypergeometric in n and k so is $p(n)F(n, k) - q(n)F(n+1, k)$, so considering the latter as a function of k we can apply Gosper's algorithm to find a function $G(n, k)$ such that

$$p(n)F(n, k) - q(n)F(n+1, k) = G(n, k+1) - G(n, k).$$

If such a function $G(n, k)$ indeed exists, then the proof of (6) would follow by summing w.r.t. k.

Example. For Dixon's classical identity (Knuth, 1973, ex. 1.2.6.62),

$$F(n, k) = (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k}, \qquad R(n) = \frac{(n+b+c)!}{n!b!c!}.$$

Hence $q(n) = (n+1)$, $p(n) = (n+b+c+1)$, and the $G(n, k)$ that does the job is $((k+b) \times (k+c)/2(n-k+1))F(n, k)$.

It turns out that all Gosper's conjectured hypergeometric identities (and all those in Bailey (1935)) can be proved by the above method. But *WHY*? This will be explained in the next section, where we continue where we left off at section 2, and where we show that a suitable modification of the above method is always guaranteed to work. The reason it did not occur to Gosper to apply the above simple method is that a priori there is no reason to believe that such a closed form $G(n, k)$ exists. It occurred to me because it followed naturally from the general theory of Zeilberger (preprint). This is another example of the importance of a general theory. In addition to supplying formal proofs of validity for algorithms that seem to work, it, even more importantly, inspires and suggests new algorithms.

4. Why?

Thanks to the general theory of Zeilberger (preprint), we got that for any holonomic $F(n, k)$ we have a function $G(n, k)$ that is given by (4), such that

$$S(N, n)F(n, k) = G(n, k+1) - G(n, k). \tag{3}$$

Because of (4) it also follows that $G(n, k)$ is holonomic as well. What happens if $F(n, k)$ is not only holonomic but in fact closed form? The crucial observation is

LEMMA. *If $F(n, k)$ is of closed form and $R(N, K, n, k)$ is any linear partial difference operator with polynomial coefficients, then $G(n, k) := R(N, K, n, k)F(n, k)$ is also of closed form, and in fact is a multiple of $F(n, k)$ by a certain rational function.*

PROOF. Since $F(n+1, k)/F(n, k)$ and $F(n, k+1)/F(n, k)$ are both rational functions in (n, k), it follows by induction that for any integers r and s, $F(n+r, k+s)/F(n, k)$ is also a rational function. But $(R(N, K, n, k)F(n, k))/F(n, k)$ is a linear combination of terms of the form $p_{r,s}(n, k)F(n+r, k+s)/F(n, k)$, where the $p_{r,s}$ are polynomials. Since the sum of rational functions is again rational, it follows that indeed $G(n, k)/F(n, k)$ is a rational function. Since any rational function is *ipso facto* of closed form, and the product of closed form functions is again closed form, it follows that $G(n, k)$ is of closed form.

Because of the above lemma, we now know that there exists a linear recurrence operator $S(N, n)$ and a *closed form* $G(n, k)$ such that (2) is true. **If we knew $S(N, n)$ beforehand then Gosper's algorithm,** (w.r.t. k) **can be used to find $G(n, k)$.** The problem is that many times we do not know $S(N, n)$ beforehand. Now that we have the theoretical confidence that such an $S(N, n)$ exists, we can try and conjecture it empirically, by trying a generic equation, plugging in values and solving the resulting equations. However, we do not need to do it! Gosper's algorithm can be extended to find $S(N, n)$ as described in the next section.

5. An Algorithm for Definite Hypergeometric Summation using Gosper's Algorithm for Indefinite Hypergeometric Summation

Suppose you know, or expect, that $S(N, n)$ has order L. Write $S(N, n)$ in generic form

$$S(N, n) := \sum_{i=0}^{L} s_i(n)N^i.$$

Now express $S(N, n)F(n, k)$ in terms of these generic $s_i(n)$ and carry Gosper's algorithm (Gosper, 1978) w.r.t. k. The "ground field" is no longer the field of rational numbers, but rather the field of rational functions in n. Now another miracle happens. All the s_i occur linearly in the polynomial $p(k)$ in Gosper's algorithm! In Gosper's algorithm, everything boils down to trying to find a *polynomial* $f(k)$ such that ([8] in Gosper, 1978)

$$p(k) := q(k+1)f(k) - r(k)f(k-1), \tag{8}$$

where $p(k)$, $q(k)$, and $r(k)$ are certain polynomials obtained from the input. Gosper then easily finds an upper bound for the degree of $f(k)$, obtained from the "input" polynomials p, q, r, writes $f(k)$ in generic form, substitutes in (8), compares coefficients of powers of k, gets a system of linear equations in the unknown coefficients of $f(k)$, and solves them. If he obtains a solution, then it means that indeed (8) can be solved, and from the $f(k)$ he gets the anti-difference.

If we knew $S(N, n)$, i.e. the $s_i(n)$ beforehand, then we just apply Gosper's algorithm. But the trouble is that we do not know the $s_i(n)$ to begin with, and they are parts of our unknowns. In fact we want to find $s_i(n)$ such that $S(N, n)F(n, k)$ is indefinitely summable w.r.t. k. Following Gosper's algorithm with the generic, as yet unknown, $s_i(n)$, it is easy to see that the expression for $p(k)$, that features in (8), is a linear combination of the $s_i(n)$, with coefficients that are polynomials in k (whose coefficients, in turn, are rational functions of n). We are looking for those $s_i(n)$ for which the system produced by Gosper's algorithm will be solvable. The resulting system is thus a system of equations both in the coefficients of $f(k)$ and the $s_i(n)$. If there is a solution, this means that with the given L there is indeed an operator $S(N, n)$ of order L, and a $G(n, k)$ of closed form such that (3) holds, and the algorithm gives both of them. If there is no solution, this means that no such L exists, and we have to try $L+1$. By the general theory, we know that eventually we will succeed. Furthermore, it is easy to get an a priori upper bound for L.

6. Implementation

The most time- and space-consuming part of Gosper's algorithm is the first step. Recall that the first step in Gosper's algorithm is writing the summand a_k/a_{k-1} in the form

$$\frac{a_k}{a_{k-1}}=\frac{p(k)}{p(k-1)}\frac{q(k)}{r(k)}, \tag{9}$$

where $p(k)$, $q(k)$, $r(k)$ are polynomials in k subject to the following condition:

$$GCD(q(k), r(k+j))=1, \text{ for } \textit{all non-negative integers } j.$$

This is the same as finding the polynomial of the largest possible degree, $p(k)$, such that a_k can be written as

$$a_k=p(k)b_k,$$

such that b_k is a hypergeometric sequence that is "simpler" in the sense that b_k/b_{k-1} is simpler than a_k/a_{k-1}.

When Gosper's algorithm is used as a subroutine in the present algorithm, the "a_k" of Gosper is already given in the form $p_1(k)b_k$, for some polynomial $p_1(k)$. It is therefore wise to leave this polynomial part alone, and try to find the representation (9) for b_k alone.

$$\frac{b_k}{b_{k-1}}=\frac{p_2(k)}{p_2(k-1)}\frac{q(k)}{r(k)},$$

and then $p(k)=p_1(k)p_2(k)$.

Another nice feature is that when we enter Gosper's subroutine, initially the "non-polynomial part" of a_k, is given as a quotient of products of factorials. Thus the initial polynomials $q(k)$, $r(k)$, when we enter the first step, are already factorized into linear factors. We should be very careful to keep everything in factored form, and never expand the polynomials until the very end. To achieve this, it is better to store such a polynomial as a list of its linear factors, and to operate on lists whenever possible.

A listing of a MAPLE program implementing the algorithm is given in Zeilberger (submitted). There we assume that the input $F(n, k)$ is given in the form

$$F(n, k)=\frac{\prod_{i=1}^{M}(\alpha_i n+\beta_i k+c_i)!}{\prod_{i'=1}^{M'}(\alpha'_{i'} n+\beta'_{i'} k+c'_{i'})!}x^k, \tag{10}$$

where α_i, β_i, $\alpha'_{i'}$, $\beta'_{i'}$, are specific integers, while c_i, c'_i, x are any expressions that do not depend on n and k.

The a_k that is inputted into Gosper's algorithm is

$$S(N, n)F(n, k)=\sum_{i=0}^{L} s_i(n)F(n+i, k).$$

We write it as

$$\left[\sum_{i=0}^{L}(s_i(n)F(n+i, k)/F(n, k))\right]F(n, k).$$

Each term in the above sum is found in factored form, and when adding, the numerator has to be expanded, but the denominator can be still kept in factored form. The numerator of the above sum is part of the "initial $p_1(k)$" discussed above, and does not have to be considered in the first step of Gosper's subroutine any more. Since everything else is given in factored form, getting the final decomposition (9) can be carried very fast.

7. Applications

7.1. PROVING IDENTITIES

If the outputted recurrence $S(N, n)$ turns out to be first order, i.e. $L=1$ already works, then we have discovered an identity, since a first order recurrence $a_0(n)R(n)+a_1(n)R(n+1)=0$ can always be solved in closed form. In very rare cases $R(n)$ has closed form although $S(N, n)$ obtained by the algorithm is of higher order. Even in this case it is possible to prove the conjectured identity. Simply check if $R(n)$ is indeed a solution of $S(N, n)R(n)\equiv 0$, and check the required number of initial conditions.

Example 1 of Zeilberger (submitted) shows how the program proved the venerable Dougall identity (Bailey, 1935). It can equally well prove all other terminating hypergeometric sum identities in Bailey's classic book, as well as all of Gosper's "strange" identities considered in Gessel & Stanton (1982).

Since most non-terminating hypergeometric identities follow from terminating ones either by using Carlson's theorem (Bailey, 1935), or as limiting cases, the algorithm is also useful in this case.

Ira Gessel pointed out to me how my algorithm can be used to discover (and of course, *prove* at the same time) new identities. Start with an $F(n, k)$ with a certain number of free parameters (the c_i, $c'_{i'}$, and x of (10)). Let the program output the $(L+1)$-term recurrence, and then equate $L-1$ of these coefficients to 0, solving the resulting equation in the c_i, $c'_{i'}$ and x, thus finding sets of values of the parameters for which the $(L+1)$-term recurrence reduces to a two-term recurrence.

Computer-generated proofs, combined with human ingenuity and insight, sometimes lead to interesting theoretical advances. In Wilf & Zeilberger (to appear) (two papers) Herbert Wilf and I show how the computer-generated proofs that were obtained by the present program lead to the natural concepts of *WZ pair* and *dual identity*.

7.2. PROVING TRANSFORMATION FORMULAS FOR HYPERGEOMETRIC SERIES

The format of a transformation formula is

$$\sum_k F_1(n, k) = \sum_k F_2(n, k),$$

where both $F_1(n, k)$ and $F_2(n, k)$ are hypergeometric in (n, k), and we assume that both series are terminating. The algorithm can be used to find the recurrences satisfied by either side, and then one checks whether these are the same, or at any rate "equivalent" (see Zeilberger, preprint; to appear). It then follows that the identity is true provided it is true for the first few initial values of n.

Example 2 of Zeilberger (submitted) contains the input that yields the computer-generated proof of the important Whipple transformation (Bailey, 1935, p. 25).

7.3. A TOOL FOR THE ORTHOGONAL POLYNOMIALS HUNTER

Hitherto, given a set of polynomials of hypergeometric type, it took considerable human ingenuity and stamina to prove that these indeed constitute a set of orthogonal polynomials. This can now be done with the present program. If the recurrence obtained by the program has the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x),$$

then, as is well known, (e.g. Chihara, 1978) the $\{p_n(x)\}$ do indeed form a set of orthogonal polynomials. Example 3 of Zeilberger (submitted) gives the input with which the program proved that the celebrated Wilson polynomials (Wilson, 1980), that contain all the classical families of orthogonal polynomials as special or limiting cases, are indeed orthogonal.

7.4. PROVING THAT AN IMPORTANT SEQUENCE SATISFIES AN IMPORTANT RECURRENCE

I have already mentioned Apery's proof of the irrationality of $\zeta(3)$ that required that

$$a(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the recurrence

$$(n+1)^3 a(n+1) - (34n^3 + 51n^2 + 27n + 5)a(n) + n^3 a(n-1) = 0.$$

Van der Poorten (1979, p. 200) writes: "Neither Cohen nor I had been able to prove [the recurrence] in the ensuing 2 months". Equipped with my program, it would have taken them a few seconds.

7.5. YOU NEVER KNOW WHEN YOU WILL ENCOUNTER A BINOMIAL COEFFICIENTS SUM

Although sums of products of binomial coefficients (alias terminating hypergeometric sums) "are not quite everywhere dense in combinatorial problems" (Askey, 1975, p. 94, lines 8 and 7 from the bottom), they do occur in the most unexpected places, even outside of combinatorics proper.

Gregory Chaitin (1987), in his celebrated concretization of Godel's incompleteness proof and Hilbert's 10th problem, needed the number of so-called (LISP) S-expressions. He easily found an expression for it in terms of a certain binomial coefficients sum, but then used ad hoc methods to find a recurrence (Chaitin, 1987, pp. 171-174). The present program does it in a few seconds. (See example 4 in Zeilberger, submitted.)

7.6. A CONSIDERABLE SAVING IN COMPUTATION

Many times it is required to compile a table of a sequence defined by

$$R(n) := \sum_k F(n, k).$$

If a table of the values for $0 \le n \le N$ is required, and $R(n)$ is computed straight from the definition, then one needs $O(N^2)$ operations. However, if one has a recurrence, then one only needs $O(N)$ operations. An example that arose in probability theory is given in Wimp & Zeilberger (1989).

7.7. A USEFUL TOOL FOR ASYMPTOTICS

Knowing the recurrence that a sequence satisfies is also useful for finding the asymptotics. The *Birkhoff–Trijinski* method that was exposited in Wimp & Zeilberger (1985), inputs a recurrence and outputs the asymptotics of the dominant solution. Although there are standard techniques (see Knuth, 1973*b*, pp. 66–67) for *positive* sums of products of binomial coefficients, these fail completely for *alternating* sums. With the present method, it is a routine matter to find the recurrence, and by using the Birkhoff–Trijinski method it is then possible to find the asymptotics.

7.8. ONE MORE EXAMPLE, COURTESY OF IRA GESSEL

Ira Gessel, in an electronic message, described to me an application he found to *Baxter permutations.* I reproduce his message intact:

> I did find one nice application that might be worth adding to your list. In "The number of Baxter permutations" by F. Chung, R. Graham, V. Hoggatt and M. Kleiman, in JCT A 24 (1978), 382–394, they show that the number $B(n+1)$ of "Baxter permutations" on $1, 2, \ldots, n+1$ is $3F2(-n, -n-1, -n-2; 2, 3; -1)$ and they give without proof a complicated 4-term recurrence which they attribute to Paul Bruckner. When I saw the formula I realized that it could be transformed by a quadratic transformation to a $3F2(1)$, and therefore a 3-term recurrence had to exist. Using the $3F2(1)$ contiguous relations given in a paper of Jim Wilson's I computed the 3-term recurrence, but apparently I made a mistake, since it didn't work. But your program very quickly came up with the recurrence
>
> $$(-8(n+2)(n+1) + (-7n^2 - 49n - 82)N + (n+6)(n+5)N^2)S(n) = 0,$$
>
> where $S(n) = B(n+1)$.

8. What's Next?

A continuous and a q-analog of this paper are given in Almkvist & Zeilberger (1990) and Zeilberger (in preparation) respectively. In particular the program of Zeilberger succeeded in proving (a generalization of) the famous Rogers–Ramanujan identities [Ekhad & Tre (to appear)]. So far we can only handle single sums. The next step would be to develop fast algorithms for multisums, in which case the slow algorithms are even slower, and in fact, intractable.

I wish to thank Gert Almkvist, Richard Askey, Dominique Foata, George Gasper, Ira Gessel, Don Knuth, Dennis Stanton, Herb Wilf, and Jet Wimp, for many stimulating conversations and/or communications. I also thank Shalosh B. Ekhad and Sol Tre for computational help, and Viggo Kann for helpful programming tips.

The brilliant observation that, in (3), $R(N, K, n, k)$ may depend on k, was made by Gert Almkvist.

References

Almkvist, G., Zeilberger, D. (1990). The method of differentiating under the integral sign. *J. Symbolic Computation* **10,** 571-591.

Askey, R. (1975). Orthogonal polynomials and special functions. *Regional Conference Series in Applied Mathematics* **21,** SIAM, Philadelphia.

Bailey, W. N. (1935). Generalized hypergeometric series. *Cambridge Math. Tracts* **32,** London: Cambridge University Press. (Reprinted: Hafner, New York, 1964.)

Bernstein, I. N. (1971). Modules over a ring of differential operators, study of the fundamental solutions of equations with constant coefficients. *Functional Analysis and Its Applications* **5,** (2), Russian original: 1-16, English translation: 89-101.

Bjork, J.-E. (1979). *Rings of Differential Operators.* Amsterdam: North-Holland.

Chaitin, G. (1987). *Algorithmic Information Theory.* Cambridge: Cambridge University Press.

Chihara, T. S. (1978). *An Introduction to Orthogonal Polynomials.* New York: Gordon and Breach.

Dyson, F. (1972). Missed opportunities. *Bull. Amer. Math. Soc.* **78,** 635-653.

Ekhad, S. B., Tre, S. (to appear). A purely verification proof of the Rogers-Ramanujan identity. *J. Comb. The. A.*

Gessel, I., Stanton, D. (1982). Strange evaluations of hypergeometric series. *SIAM J. Math. Anal.* **13,** 295-308.

Gosper, R. W., Jr. (1978). Decision procedure of indefinite summation. *Proc. Natl. Acad. Sci. USA* **75,** 40-42.

Graham, R., Patashnik, O., Knuth, D. E. (1989). *Concrete Mathematics.* Reading: Addison Wesley.

Knuth, D. E. (1973*a*). Fundamental Algorithms, In: *The Art of Computer Programming,* Vol. 1, Reading: Addison-Wesley.

Knuth, D. E. (1973*b*). Sorting and searching. In: *The Art of Computer Programming,* Vol. 3. Reading: Addison-Wesley.

Lafon, J. C. (1983). Summation in finite terms. In: (Buchberger, B., Collins, G. E., Loos, R. eds.) *Computer Algebra Symbolic and Algebraic Computation,* 2nd edn. Vienna: Springer-Verlag.

van der Poorten, A. (1979). A proof that Euler missed..., Apery's proof of the irrationality of $\zeta(3)$. *Math. Intel.* **1,** 195-203.

Wilson, J. A. (1980). Some hypergeometric orthogonal polynomials. *SIAM J. Math. Anal.* **11,** 690-701.

Wilf, H., Zeilberger, D. (to appear). Rational functions certify combinatorial identities. *J. Amer. Math. Soc.*

Wilf, H., Zeilberger, D. (to appear). Towards computerized proofs of identities. *Bull. Amer. Math. Soc.*

Wimp, J., Zeilberger, D. (1985). Resurrecting the asymptotics of linear recurrences. *J. Math. Ana. Appl.* **111,** 162-177.

Wimp, J., Zeilberger, D. (1989). How likely is Polya's drunkard to stay in $x \geq y \geq z$? *J. Stat. Physics* **57,** 1129-1135.

Zeilberger, D. (to appear). A holonomic systems approach to special function identities. *J. Comp. Appl. Math.*

Zeilberger, D. (1990). A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.* **80,** 207-211.