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A ONE-LINE HIGH SCHOOL ALGEBRA PROOF OF THE UNIMODALITY OF THE GAUSSIAN POLYNOMIALS $\begin{bmatrix} n \\ k \end{bmatrix}$ FOR $k < 20$*

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Abstract. By "squeezing the combinatorics out" of Kathy O'Hara's magnificent combinatorial proof of the unimodality of the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$, we give an extremely short and elementary proof of this unimodality for $k <$ any *fixed* integer, and a fairly short, semi-combinatorial proof for general k. Combined with Macdonald's paper in this volume the present paper implies an entirely elementary algebraic proof of the unimodality of the Gaussian polynomials.

1. Introduction and Preliminaries. Kathy O'Hara ([2],[3], see also [9]) has recently astounded the world of combinatorics by giving the long-sought-for combinatorial proof of the unimodality of the "Gaussian polynomials"

$$G(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix} = \frac{(1-q^{n+1})(1-q^{n+2})\dots(1-q^{n+k})}{(1-q)(1-q^2)\dots(1-q^k)}.$$

Prior to O'Hara's proof there were only indirect proofs that made use of very advanced mathematics. The reader is referred to Proctor's elegant paper [8] for the history and significance of this result. A polynomial $a_0 + \cdots + a_N q^N$ is *unimodal* if it is increasing up to a point and then it is decreasing, i.e. there exists an index i such that $a_0 \le a_1 \le \cdots \le a_i \ge \cdots \ge a_N$.

The first proof of the unimodality of the Gaussian polynomials was given by Sylvester[7], as a consequence of a deep theorem in the classical theory of invariants. Among the many other proofs we only mention White's[8] elegant Polya theoretic proof, Macdonald's[4] (I.8. ex 4) "functorial" proof and the linear algebra proofs of Proctor[5] and Stanley[6].

A careful scrutiny of O'Hara's proof enabled me to "algebrize" her combinatorial proof to get extremely short, though unmotivated, high school algebra proofs of the unimodality of $G(n,k)$ for $k \le A$, where A is a fixed number that depends on the size of one's computer. In order to prove the general result I still need to use part of O'Hara's combinatorial argument.

The *darga* of a polynomial $p(q) = a_i q^i + \cdots + a_j q^j$, with $a_i \neq 0$ and $a_j \neq 0$, is defined to be $i + j$, i.e. the sum of its lowest and highest powers. For example *darga* $(q^2 + 3q^3) = 5$ and *darga* $(q^2) = 4$. A polynomial $p(q) = a_g q^g + \cdots + a_{m-g} q^{m-g}$ is *symmetric* if $a_i = a_{m-i}$ for every i.

We need to make three simple observations.

Observation 1. The sum of two symmetric and unimodal polynomials of darga m is also symmetric and unimodal of darga m. □

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Observation 2. The product of two symmetric and unimodal polynomials of dargas m and m' respectively is a symmetric and unimodal polynomial of darga $m+m'$.

Proof. It is readily seen that a polynomial of darga m is symmetric and unimodal if and only if it can be expressed as a sum of "atomic" entities of the form $c(q^r+q^{r+1}+\cdots+q^{m-r})$, for some *positive* constant c and some integer $r \le m/2$. So it is enough to prove that the product of two such atoms of dargas m and m' is unimodal. But this is clear because

$$(q^r+\dots q^{m-r})(q^{r'}+\dots+q^{m'-r'}) = q^{r+r'}+2q^{r+r'+1}+\dots+2q^{m+m'-r-r'-1}+q^{m+m'-r-r'}.$$

□

Observation 3. If p is symmetric and unimodal of darga m then $q^\alpha p$ is symmetric and unimodal of darga $m+2\alpha$. □

In order to make part of this paper comprehensible to *real* high school students we will first spell out the proofs of the unimodality of $G(n,k)$ for $k \le 6$. We will have one routinely verifiable line for each of $k = 1,\dots,6$. Later we will see that all these lines are just special cases of a single line, that is however long and complicated. This hairy line ((KOH) below) is an amazing q-binomial identity, that is a consequence of O'Hara's combinatorial proof. In order to prove this amazing identity for *every* k, we will have to make use of O'Hara's argument. However for $k \le$ *any fixed number*, this q-identity is nothing but a *finite* algebraic identity, easily verifiable by MAPLE, MACSYMA and their like.

2. Proof Of The Unimodality Of G(n, k) For $k \le 6$. In the following proofs of Propositions $k, 2 \le k \le 6$, one should prefix the following sentence: "By the inductive hypothesis, propositions i for $i < k$, and the three observations, the unimodality of $G(n,k)$ follows from the following routinely verifiable algebraic identity (both sides are polynomials in q^m with coefficients that are rational functions of q)."

PROPOSITION 1. *$G(m,1)$ is symmetric and unimodal of darga m.*

Proof. $G(m,1) = 1+q+q^2+\cdots+q^m$.

PROPOSITION 2. *$G(m,2)$ is symmetric and unimodal of darga $2m$.*

Proof. $G(m,2) = q^2G(m-2,2)+G(2m,1)$.

PROPOSITION 3. *$G(m,3)$ is symmetric and unimodal of darga $3m$.*

Proof. $G(m,3) = q^6G(m-4,3)+q^2G(2m-2,1)G(m-2,1)+G(3m,1)$.

PROPOSITION 4. *$G(m,4)$ is symmetric and unimodal of darga $4m$.*

Proof.

$$G(m,4) = q^{12}G(m-6,4)+q^6G(2m-4,1)G(m-4,2)+ q^4G(2m-4,2)+q^2G(3m-2,1)G(m-2,1)+G(4m,1).$$

PROPOSITION 5. *$G(m,5)$ is symmetric and unimodal of darga $5m$.*

Proof.

$$G(m,5) = q^{20}G(m-8,5)+q^{12}G(2m-6,1)G(m-6,3)+q^6G(3m-4,1)G(m-4,2)+ q^8G(2m-6,2)G(m-4,1)+q^2G(4m-2,1)G(m-2,1)+q^4G(3m-4,1)G(2m-4,1)+ G(5m,1).$$

PROPOSITION 6. *$G(m,6)$ is symmetric and unimodal of darga $6m$.*

Proof.

$$G(m,6) = q^{30}G(m-10,6)+q^{20}G(m-8,4)G(2m-8,1)+q^{14}G(m-6,2)G(2m-8,2)+ q^{12}G(2m-8,3)+q^{12}G(m-6,3)G(3m-6,1)+q^8G(m-4,1)G(2m-6,1)G(3m-6,1)+ q^6G(3m-6,2)+q^6G(m-4,2)G(4m-4,1)+q^4G(2m-4,1)G(4m-4,1)+q^2G(m-2,1)+ G(5m-2,1)+G(6m,1).$$

3. An Amazing q-Binomial Identity That Implies The Unimodality Of G(n, k) For Every n and k.

$$G(n,k) = \qquad (KOH)$$

$$\sum_{\{(d_1,\dots,d_k)\ ;\ \sum_{i=1}^k id_i = k\}} q^{k(\sum_{i=1}^k d_i)-k-\sum_{1\le i<j\le k}(j-i)d_id_j} \prod_{i=0}^{k-1} G((k-i)n-2i+ \sum_{j=0}^{i-1} 2(i-j)d_{k-j}, d_{k-i})$$

Note that the sum here is over all partitions of $k : i^{\{d_i\}}$, where a^b means "a repeated b times". Thus, for a fixed k, the number of summands is $p(k)$, the number of partitions of k, which is asymptotically roughly (by the Hardy-Ramanujan formula [1]) $e^{c\sqrt{k}}$, for some constant c. Each summand is a product of at most $\sqrt{(2k)}$ terms. The special cases $2 \le k \le 6$ were given above. For every specific k this identity is a routinely verifiable identity, but of course since $p(k)$ grows rapidly it soon becomes impractical.

As I have already mentioned, I obtained (KOH) by "algebrizing" O'Hara's [3] main result, as simplified in [9]. However I was unable to completely algebrize her proof, so the proof of (KOH) will have to be combinatorial, using O'Hara's argument. I am offering twenty five dollars for an elementary, non-combinatorial, proof of (KOH), whose length is not to exceed 2 pages. [Note added in the revised version: Ian Macdonald has won this prize, see his paper [10] in the present volume.]

The proof of (KOH) will be given in the next section, but before let us show why (KOH) implies the unimodality of $G(n,k)$. By the inductive hypothesis, $G(a,b)$ is symmetric and unimodal of darga ab, for $a < n$ or $b < k$.

The only $G(a,b)$ on the right side of (KOH) for which $b = k$ is the one corresponding to $d_1 = k$, $d_i = 0$ for $1 < i \leq k$, and this term is $q^{k(k-1)}G(n-2(k-1),k)$, which by the natural inductive hypothesis and observations 2 and 3 is symmetric and unimodal of darga nk. All the other terms have $d_i < k$ and so the $G(a,b)$'s featuring there have $b < k$. Now it follows by the inductive hypothesis and observations 2 and 3 that each term is symmetric and unimodal of a certain darga, and a straightforward calculation shows that the power of q that appears is just the right one to make each term have darga nk. The rest follows from Observation 1.

4. Proof Of (KOH). Let

$$U(b,a) := \{p = (p_1,\ldots,p_a); 0 \leq p_1 \leq \cdots \leq p_a \leq b\}.$$

For every p in $U(b,a)$ let

$$\text{weight } (p) := q^{p_1+\cdots+p_a},$$

and for any subset S of $U(b,a)$ (including of course $U(b,a)$ itself) let
$\text{weight } (S) := \sum_{p \in S} \text{weight } (p)$.

It is well known (e.g. [1], 3.2) and easy to see that *weight* $(U(b,a)) = G(b,a)$.

In [9] were defined certain subsets $U(b,a;m,d)$ of $U(b,a)$ that depend on two extra parameters m and d. We also defined the subsets $\overline{U}\ (b,a;m)$ to be the union of $U(b,a;m',d')$ over all $m' \leq m$ and all d'. Let

$$v(b,a;m) := \text{weight } (\overline{U}(b,a;m))$$

Taking weight in the O'Hara Structure Theorem of [9], we get

$$\text{weight } (U(b,a;m,d)) = q^{2bd-md(d+1)}G(ma+2m-2b,d)v(b-md,a-2d;m-1),$$

and summing over all conceivable d, we get

$$v(b,a;m)-v(b,a;m-1) = \sum_{d>0} q^{2bd-md(d+1)}G(ma+2m-2b,d)v(b-md,a-2d;m-1),$$

From which easily follows

$$(*) \quad v(b,a;m) = \sum_{d\geq 0} q^{2bd-md(d+1)}G(ma+2m-2b,d)v(b-md,a-2d;m-1).$$

(KOH) is obtained by starting with $G(b,a) = v(b,a;b)$ and iterating (*) b times. The first iteration expresses $v(b,a;b)$ in terms of a single sum involving $v(-,-;b-1)$ and $G(-,-)$, the second iteration would give a double sum that feature $v(-,-;b-2)$ and $G(-,-),\ldots$, until one gets a b-fold sum that only features products of $G(-,-)$, in which point we have arrived at (KOH).

5. Remarks. Identity (KOH) would never have been discovered without O'Hara's combinatorial breakthrough. However, once discovered, it is conceivable that a non-combinatorial high-school algebra proof exists. Ian Macdonald, in the paper that follows, gives such a proof. It was noted by Ron Evans and Dennis Stanton that the identity proved by Macdonald is in fact slightly different, although it too implies the unimodality of the Gaussian polynomials. In the original (KOH) $G(n,k)$ is taken to be zero whenever n is negative, whereas in Macdonald's version [10], that Dennis Stanton named (MACKOH), $G(n,k)$ is defined by its formula $(1-q^{n+1})\ldots(1-q^{n+k})/(1-q)\ldots(1-q^k)$, for *every n*. (KOH) and (MACKOH) coincide for $n \geq k$, since then all the $G(-,-)$ appearing on the right side have non-negative n. It follows that (MACKOH) suffices to prove unimodality, since, by the symmetry of the Gaussian polynomials $G(n,k)$, we already know, by induction, that $G(n,k)$ is unimodal for $n < k$ ($G(n,k) = G(k,n)$), so we only have to use (KOH) for $n \geq k$.

(KOH) implies (MACKOH), since the later is a polynomial identity in q^n, and (KOH) testifies to its truth for an infinite number of cases. It is not known whether (MACKOH) can be used to prove (KOH).

(KOH) turned out to have some other surprising consequences. In [11] it is used to prove that the coefficients $(1-q)^{\min([(k+1)/2],[(n+1)/2])}G(n,k)$ alternate in sign whenever at least one of n and k is even. This proved and extended Andrew Odlyzko's conjecture that the MacLaurin coefficients of the reciprocal of the q-analog of $n!$ alternate in sign.

While (MACKOH) suffices for the unimodality of $G(n,k)$, it cannot be used for the above mentioned result of [11]. We still need (KOH), and consequently the only proof known at present of this result uses combinatorics.

David Bressoud [12] [13] found an elegant half combinatorial and half algebraic proof of (KOH), and in the process found a far reaching generalization.

Goodman and O'Hara discovered that the definition of 'spread' in [3] causes a minor glitch in the derivation of (KOH) given in section 4. In [14] they introduced a very minor modification in the definition of 'spread' that makes the glitch disappear.

John Stembridge informed me that he now has a purely algebraic Hall–Littlewood function proof of (KOH) and of other identities of Bressoud–Andrews–Gordon style. Furthermore he can surgically remove the intimidating Hall–Littlewood functions and what remain are high school algebra proofs.

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AN ELEMENTARY PROOF OF A q–BINOMIAL IDENTITY

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In the previous paper [Z] D. Zeilberger asks for an elementary, non–combinatorial proof of the identity (KOH). We shall give such a proof.

We shall use the notation of [M]. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition, let $|\lambda| = \sum \lambda_i$ denote the weight of λ, and λ' the conjugate partition. Then for each $i \geq 1$,

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$$

is the number of parts of λ equal to i. If $\mu = (\mu_1, \mu_2, \dots)$ is another partition, $\lambda + \mu$ denotes the partition $(\lambda_1 + \mu_1, \lambda_2, \mu_2, \dots)$ and $\lambda \cup \mu$ the partition whose parts are those of λ and μ, arranged in descending order. We have then $(\lambda \cup \mu)' = \lambda' + \mu'$.

Furthermore, for any partition λ we define

$$n(\lambda) = \sum (i-1)\lambda_i = \sum \binom{\lambda'_i}{2}$$

and

$$b_\lambda(q) = \prod_{i \geq 1} \varphi_{m_i(\lambda)}(q),$$

where q is an indeterminate, $\varphi_m(q) = (1-q)(1-q^2)\dots(1-q^m)$ if $m \geq 1$, and $\varphi_0(q) = 1$. Then the identity (KOH) in [Z] may be written in the form

$$\text{(KOH)} \qquad G(N,k) = \sum_{|\lambda|=k} q^{2n(\lambda)} F(N,\lambda)$$

where

$$F(N,\lambda) = \prod_{i \geq 1} G((N+2)i - 2\sum_{j=1}^{i} \lambda'_j, m_i(\lambda))$$

and $G(N,k)$ is the gaussian polynomial (or q–binomial coefficient)

$$G(N,k) = \begin{bmatrix} N+k \\ k \end{bmatrix} = \frac{(1-q^{N+1})\cdots(1-q^{N+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

The identity (KOH) is a consequence of the following two identities:

$$\text{(A)} \qquad G(N,k) = \sum_{r=0}^{k} q^{Nr}/\varphi_r(q^{-1})\varphi_{k-r}(q),$$

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