PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 1, September 1989

THE ODLYZKO CONJECTURE AND O'HARA'S UNIMODALITY PROOF

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(Communicated by Andrew M. Odlyzko)

ABSTRACT. We observe that Andrew Odlyzko's conjecture that the Maclaurin coefficients of $1/[(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})]$ have alternating signs is an almost immediate consequence of an identity that is implied by Kathy O'Hara's recent magnificent combinatorial proof of the unimodality of the Gaussian coefficients.

To a true combinatorialist, a combinatorial result is not properly proved until it receives a *direct combinatorial proof*. This is why Kathy O'Hara's long-sought-for constructive proof of the unimodality of the Gaussian polynomials ([4], [5], see also [6]) generated so much excitement in combinatorial circles. However to non-combinatorialists, a direct combinatorial proof is "just another proof." O'Hara's proof is longer than most of the dozen previous proofs, and probably would not add any insight to anyone who is not a genuine combinatorialist. Moreover, it does not seem to be generalizable at first sight. Yet it turned out to imply a deep result (KOH) to which hitherto there was no known proof of any kind.

In this note we shall prove and generalize a conjecture of Odlyzko, using O'Hara's result. Odlyzko's results imply that for k sufficiently large, the first k coefficients in

$$\frac{1}{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})}=\frac{(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)}$$

alternate in sign. He conjectured that in fact for every $k \geq 0$, all of the coefficients of the above series alternate in sign. We prove the sharper result

Theorem 1. *For any integer* k,

$$\frac{(1-q)^{[(k+1)/2]}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

has coefficients which alternate in sign.

Received by the editors March 28, 1988 and, in revised form, November 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05A17, 05A20, 05A30, 10A45.
The first author's work was partially supported by NSF grant DMS:8700995.
The second author's work was partially supported by NSF grant DMS:8800663.

Note that the exponent of $(1-q)$ is best possible, since if $[(k+1)/2]$ is replaced by $[(k-1)/2]$ then the pole $q=1$ has the highest order among all the poles, all of which are roots of unity, so a partial fraction expansion would yield that the coefficients are asymptotically of the same sign.

Odlyzko has informed the authors that Theorem 1 can be used to shorten the proof in [3] by at least one third.

We will prove a more general result. Recall that the Gaussian polynomials are defined for nonnegative integers k and n by

$$\text{(GP)} \qquad G(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

If n is negative, we put $G(n,k)=0$. We will prove:

Theorem 2. *For nonnegative integers n and k, with nk even, $G(n,k)(1-q)^m$ has coefficients which alternate in sign, where $m = \min\{[(k+1)/2],\ [(n+1)/2]\}$.*

Theorem 1 follows from Theorem 2 upon taking n even and letting $n \to \infty$.

Theorem 2 will follow from the following amazing q-binomial identity that was derived in [7], by "algebrizing" O'Hara's main theorem ([4], [5], [6]).

$$\text{(KOH)} \qquad G(n,k) = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} G\Big((k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j},\ d_{k-i}\Big),$$

where

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

The sum in (KOH) is over all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of k. The integer d_i is the multiplicity of i in λ, thus in frequency notation $\lambda = 1^{d_1}2^{d_2}\cdots i^{d_i}\cdots$. In this notation,

$$2n(\lambda) = \sum_{i=1}^{k} (D_i^2 - D_i)$$

where

$$D_r = \sum_{i=r}^{k} d_i.$$

Proof of Theorem 2. By symmetry in n and k, we may assume that n is even. We proceed by induction on n and k. Theorem 2 clearly holds for $n=0$ and $k=1$.

Let

$$F(n,k) := (1-q)^{[(k+1)/2]} G(n,k).$$

Then (KOH) can be rewritten as
(KOH$'$)

$$F(2n,k) = \sum_{\lambda \vdash k} (1-q)^{\alpha(\lambda)} q^{2n(\lambda)} \prod_{i=0}^{k-1} F\Big(2(k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j},\ d_{k-i}\Big).$$

where

$$\alpha(\lambda) := m - \sum_{i=1}^{k} [(d_i + 1)/2].$$

Suppose we show that $\alpha(\lambda) \geq 0$. If $d \neq 1^k$, then each F on the right side of (KOH$'$) has a second argument less than k. If $\lambda = 1^k$, the first argument of F is less than $2n$. Thus by induction each F is alternating. Since $(1-q)^{\alpha(\lambda)}$ is alternating, and the power of q is even, the left side must be alternating. So it remains to verify that $\alpha(\lambda) \geq 0$.

First suppose that $n \geq [(k+1)/2]$, so $m = [(k+1)/2]$. Then we will show that for any partition λ of k, we have the inequality

$$(*) \qquad [(k+1)/2] - \sum_{i=1}^{k} [(d_i + 1)/2] \geq 0.$$

It is easy to see that $(*)$ is

$$[(k+1)/2] - (\text{number of parts of } \lambda + \text{number of } i \text{ with } d_i \text{ odd})/2.$$

This is nonnegative, since any part $i > 1$ of λ can contribute at most one i which has d_i odd.

Next suppose that $n < [(k+1)/2]$, so $m = n$. First we show

$$(**) \qquad n + 1 - \sum_{i=1}^{k} d_i \geq 0$$

for all partitions λ of k which occur in (KOH$'$). The key observation is that F is zero if the first argument is negative. Thus, taking the $i = k-1$ term in (KOH$'$), we see that

$$2n - 2(k-1) + \sum_{j=0}^{k-2} 2(k-1-j) d_{k-j} \geq 0,$$

which is equivalent to

$$\sum_{j=2}^{k} (j-1) d_j \geq k - 1 - n,$$

or

$$k = \sum_{j=1}^{k} j d_j \geq k - 1 - n + \text{number of parts of } \lambda.$$

The final inequality implies that λ has at most $n+1$ parts, which is $(**)$. Clearly $\alpha(\lambda) \geq 0$ holds unless λ has $n+1$ distinct parts, in which case $\alpha(\lambda) = -1$. In this case the $i = k-1$ term in (KOH$'$) is alternating ($G(0,1) = 1$) without the factor of $(1-q)$, so it is enough to prove that $\alpha(\lambda) + 1 \geq 0$. □

Remarks. To prove Theorem 1 we need only the $n \to \infty$ case of (KOH). John Stembridge rediscovered an identity of Hall which implies this result:

$$\text{(JS)} \qquad \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{d \vdash k} q^{2n(d)} \begin{bmatrix} n+1 \\ d_1, \ldots, d_k \end{bmatrix}_q .$$

Then George Andrews observed that (JS) is nothing but an iteration of q-Vandermonde. Subsequently John Stembridge and Jim Joichi gave bijections that prove (JS). Their proofs are closely related to [1].

If nk is odd, Theorem 2 cannot hold, because the leading term has the wrong sign. The exponent in Theorem 2 is not always best possible: $G(11,6)(1-q)^2$ alternates in sign.

Ron Evans has made the following related conjecture. He has verified it for $a=1$ from Theorem 2.

Conjecture. *Let n, k, and a be nonnegative integers, with $k>3$ and a odd. Let $G(n,k,a)$ be defined by (GP), with q^a replacing q in the numerator. Then the coefficients of $G(n,k,a)(1-q)^{[(k+1)/2]}$ alternate in sign if nk is even, and the coefficients of $G(n,k,a)(1-q)^{[(k+1)/2]}/(1-q^2)$ alternate in sign if nk is odd.*

Some other remarks about (KOH) can be found in [7].

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