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A PROOF OF THE G_2 CASE OF MACDONALD'S ROOT SYSTEM–DYSON CONJECTURE*

DORON ZEILBERGER†

Dedicated to Joe Gillis on the occasion of his 75th birthday

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We will prove the following theorem.

THEOREM. *Let m and n be integers and x, y and z commuting indeterminates; then the constant term of the Laurent polynomial*

$$F(x,y,z)=\left[\left(1-\frac{x}{y}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{z}{x}\right)\right]^m\left[\left(1-\frac{xy}{z^2}\right)\left(1-\frac{xz}{y^2}\right)\left(1-\frac{yz}{x^2}\right)\right]^n$$
$$\cdot\left[\left(1-\frac{y}{x}\right)\left(1-\frac{z}{y}\right)\left(1-\frac{x}{z}\right)\right]^m\left[\left(1-\frac{z^2}{xy}\right)\left(1-\frac{y^2}{xz}\right)\left(1-\frac{x^2}{yz}\right)\right]^n$$

is

$$C(m,n)=\frac{(3m+3n)!(3n)!(2m)!(2n)!}{(2m+3n)!(m+2n)!(m+n)!m!n!n!}.$$

This is the G_2 case of Macdonald's Root System–Dyson conjecture (see [6, Conjecture 2.3, and (c), p. 994]; see also Morris [7]).

Macdonald [6] showed how Selberg's integral [8] (see [1] for Aomoto's recent brilliant proof) implies his conjecture for all the so-called classical root systems. We will follow the same route and show how the G_2 case follows from a corollary of Selberg's integral that is due to Morris [7, p. 94].

After the first version of this paper was written, I was kindly informed by Dominique Foata that Laurent Habsieger [9] has independently and simultaneously obtained the results of this paper.

We only need the case $n=3$ of Morris' result that spells out to the following.

MORRIS' THEOREM ($n=3$). *Let a, b, c be integers. The constant term of the Laurent polynomial*

$$H(u,v,w;a,b,c)=[(1-u)(1-v)(1-w)]^a\quad\left[\left(1-\frac{u}{v}\right)\left(1-\frac{u}{w}\right)\left(1-\frac{v}{w}\right)\right]^c$$
$$\cdot\left[\left(1-\frac{1}{u}\right)\left(1-\frac{1}{v}\right)\left(1-\frac{1}{w}\right)\right]^b\left[\left(1-\frac{v}{u}\right)\left(1-\frac{w}{u}\right)\left(1-\frac{w}{v}\right)\right]^c$$

is

$$\frac{(a+b+2c)!(a+b+c)!(a+b)!(2c)!(3c)!}{(a+2c)!(b+2c)!(a+c)!(b+c)!a!b!c!c!}.$$

We will need the following easy corollary.

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† Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104.

COROLLARY. *The coefficient of $u^A v^A w^A$ in $H(u, v, w; a, b, c)$ above is*

$$(-1)^A \frac{(a+b+2c)!(a+b+c)!(a+b)!(2c)!(3c)!}{(a-A+2c)!(b+A+2c)!(a-A+c)!(b+A+c)!(a-A)!(b+A)!c!c!}.$$

Proof. Since $(1-t)^a(1-t^{-1})^b/t^A = (-1)^A(1-t)^{a-A}(1-t^{-1})^{b+A}$, we have

$$H(u, v, w; a, b, c)/u^A v^A w^A = (-1)^A H(u, v, w; a-A, b+A, c)$$

and taking constant terms, the corollary follows from Morris' theorem.

Finally, we need the formula shown below.

DIXON'S FORMULA (e.g. [5, 1.2.6, Ex. 62, pp. 73 and 489]). *Let M, N, K be integers; then*

$$\sum_A \frac{(-1)^A}{(M+A)!(M-A)!(N+A)!(N-A)!(K+A)!(K-A)!} = \frac{(M+N+K)!}{M!N!K!(M+N)!(M+K)!(N+K)!}.$$

To prove the theorem we let $u = x/y$, $v = y/z$, and $w = z/x$. Then $F(x, y, x) = H(u, v, w; m, m, n)$. But $uvw = 1$, so the constant term of F is the sum of all the diagonal coefficients of H. Thus by the corollary the constant term of F is

$$\sum_A (-1)^A \frac{(2m+n)!(2m+n)!(2m)!(2n)!(3n)!}{(m-A+2n)!(m+A+2n)!(m-A+n)!(m+A+n)!(m-A)!(m+A)!(n)!(n)!}$$

$$= \frac{(2m+2n)!(2m+n)!(2m)!(2n)!(3n)!}{n!n!} \cdot \sum_A \frac{(-1)^A}{(m+2n-A)!(m+2n+A)!(m+n-A)!(m+n+A)!(m-A)!(m+A)!}.$$

Using Dixon's formula with $M = m+2n$, $N = m+n$, $K = m$, we get that this is equal to

$$\frac{(2m+2n)!(2m+n)!(2m)!(2n)!(3n)!}{n!n!} \cdot \frac{(3m+3n)!}{(m+2n)!(m+n)!m!(2m+3n)!(2m+2n)!(2m+n)!}$$

$$= \frac{(3m+3n)!(3n)!(2m)!(2n)!}{(2m+3n)!(m+2n)!(m+n)!m!n!n!}. \qquad \text{Q.E.D.}$$

Since F of the theorem is obviously with integer coefficients, our theorem implies the not entirely obvious fact that $C(m, n)$ is an integer, thus solving Askey's problem [2].

The q-Analogue. We will show how Kadell's [4] recent q-analogue of Morris' theorem implies the q-analogue of the G_2 Macdonald–Dyson conjecture [6]. Since the ordinary case is just the special case $q = 1$ of the q-analogue, we could have started with the q-analogue right away, giving the ordinary reader the option to plug in $q = 1$ throughout. However we feel that this would have been very poor pedagogy. Indeed, the way mathematics is created is by slowly increasing steps of generality. Unfortunately, all too often results are presented in their overpowering full generality right from the start, thus making them very hard to read and understand, let alone use as motivation.

Let

$$(y)_a = (1-y)(1-qy)\cdots(1-q^{a-1}y)$$

and

$$[a]! = \frac{(q)_a}{(1-q)^a} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{a-1}).$$

We will prove the following theorem.

q-THEOREM. *Let m and n be integers and x, y and z commuting indeterminates; then the constant term of the Laurent polynomial*

$$F(x,y,z) = \left(\frac{x}{y}\right)_m \left(\frac{z}{y}\right)_m \left(\frac{z}{x}\right)_m \left(\frac{z^2}{xy}\right)_n \left(\frac{xz}{y^2}\right)_n \left(\frac{yz}{x^2}\right)_n \cdot \left(q\frac{y}{x}\right)_m \left(q\frac{y}{z}\right)_m \left(q\frac{x}{z}\right)_m \left(q\frac{xy}{z^2}\right)_n \left(q\frac{y^2}{xz}\right)_n \left(q\frac{x^2}{yz}\right)_n$$

is

$$\frac{[3m+3n]![3n]![2m]![2n]!}{[2m+3n]![m+2n]![m+n]![m]![n]![n]!}.$$

We need the following theorem [4].

KADELL'S q-MORRIS THEOREM ($n=3$). *Let a, b, c be integers. The constant term of the Laurent polynomial*

$$H(u,v,w;a,b,c) = (u)_a(v)_a(w)_a \left(\frac{u}{v}\right)_c \left(\frac{u}{w}\right)_c \left(\frac{v}{w}\right)_c \left(\frac{q}{u}\right)_b \left(\frac{q}{v}\right)_b \left(\frac{q}{w}\right)_b \left(q\frac{v}{u}\right)_c \left(q\frac{w}{u}\right)_c \left(q\frac{w}{v}\right)_c$$

is

$$\frac{[a+b+2c]![a+b+c]![a+b]![2c]![3c]!}{[a+2c]![b+2c]![a+c]![b+c]![a]![b]![c]![c]!}.$$

We will need the following easy corollary.

q-COROLLARY. *The coefficient of $u^Av^Aw^A$ in $H(u,v,w;a,b,c)$ above is*

$$(-1)^A q^{3A(A-1)/2} \cdot \frac{[a+b+2c]![a+b+c]![a+b]![2c]![3c]!}{[a-A+2c]![b+A+2c]![a-A+c]![b+A+c]![a-A]![b+A]![c]![c]!}.$$

Proof. We are really looking for the constant term of $H(u,v,w;a,b,c)/u^Av^Aw^A$. But since

$$\frac{(t)_a(q(1/t))_b}{t^A} = (-1)^A q^{A(A-1)/2}(q^At)_{a-A}\left(\frac{q}{q^At}\right)_{b+A}.$$

it follows that

$$\frac{H(u,v,w;a,b,c)}{u^Av^Aw^A} = (-1)^A q^{3A(A-1)/2} H(q^Au, q^Av, q^Aw; a-A, b+A, c)$$

and the corollary follows from Kadell's q-Morris Theorem.

Finally, we need the following formula.

THE q-DIXON FORMULA ([3], [5, p. 489]).

$$\sum_A \frac{(-1)^A q^{A(3A-1)/2}}{[M+A]![M-A]![N+A]![N-A]![K+A]![K-A]!} = \frac{[M+N+K]!}{[M]![N]![K]![M+N]![M+K]![N+K]!}.$$

To prove the q-Theorem we let $u = q(y/z)$, and $v = z/x$ and $w = x/y$. Then $F(x, y, z) = H(u, v, w; m, m, n)$. But $uvw = q$ so the constant term of F is the weighted sum of all the diagonal coefficients of H, where the coefficient of $u^A v^A w^A$ gets multiplied by q^A.

Thus by the corollary the constant term of F is

$$\sum_A q^A(-1)^A q^{3A(A-1)/2} \cdot \frac{[2m+2n]![2m+n]![2m]![2n]![3n]!}{[m-A+2n]![m+A+2n]![m-A+n]![m+A+n]![m-A]![m+A]![n]![n]!}$$

$$= \frac{[2m+2n]![2m+n]![2m]![2n]![3n]!}{[n]![n]!} \cdot \sum_A \frac{(-1)^A q^{A(3A-1)/2}}{[m+2n-A]![m+2n+A]![m+n-A]![m+n+A]![m-A]![m+A]!}.$$

Using the q-Dixon formula with $M = m+2n$, $N = m+n$, $K = m$, we get that this is equal to

$$\frac{[2m+2n]![2m+n]![2m]![2n]![3n]!}{[n]![n]!} \cdot \frac{[3m+3n]!}{[m+2n]![m+n]!m![2m+3n]![2m+2n]![2m+n]!}$$

$$= \frac{[3m+3n]![3n]![2m]![2n]!}{[2m+3n]![m+2n]![m+n]![m]![n]![n]!}. \qquad \text{Q.E.D.}$$

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