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NOTE

A BIJECTIVE PROOF OF CASSINI'S FIBONACCI IDENTITY

M. WERMAN
The Weizmann Institute of Science, Rehovot, Israel

D. ZEILBERGER
Department of Mathematics, Drexel University, Philadelphia, PA 19104, U.S.A.

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Cassini's Fibonacci identity ([1, 1.2.8])

$$F_{n+1}^2 - F_{n+2}F_n = (-1)^{n+1} \tag{1}$$

can be easily proved by either induction, Binet's formula, or ([1, p. 80]) by taking determinants in

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

In this paper we give a bijective proof, based upon the following combinatorial interpretation of the Fibonacci numbers.

Proposition. *Let* $A(n) = \{(a_1, \ldots, a_r); r \geqslant 0, a_i = 1$ *or* $2, a_1 + \cdots + a_r = n\}$.
We have $|A(n)| = F_{n+1}$.

Proof. $|A(0)| = 1$, $|A(1)| = 1$ and $|A(n)| = |A(n-1)| + |A(n-2)|$, since $a_r = 1$ or 2. □

Proof of (1). Let $e = (2, \ldots, 2)$; define the bijection $\pi: A(n) \times A(n) \setminus (e, e) \to A(n-1) \times A(n+1) \setminus (e, e)$ as follows. Let $[(a_1, \ldots, a_r), (b_1, \ldots, b_s)] \in A(n) \times A(n)$ and look for the first 1 in $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, \ldots$.

Case I. The first 1 is an a_k. Delete $a_k = 1$ from the first vector and insert it between b_{k-1} and b_k in the second vector.

Case II. The first 1 is a b_k, (then $a_k = 2$). Exchange a_k and b_k. □

Reference

[1] D.E. Knuth, The Art of Computer Programming, Volume 1, 2nd ed. (Addison-Wesley, Reading MA, 1973).