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Note

Some Asymptotic Bijections

EDWARD A. BENDER

Department of Mathematics, University of California at San Diego, La Jolla, California 92093

AND

DORON ZEILBERGER*

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

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The notion of an asymptotic bijection is introduced and used to give bijective proofs of infinite summation formulas for set partitions (Dobinski's formula) and involutions. © 1985 Academic Press, Inc.

1. INTRODUCTION

Suppose that $S_n \supseteq S'_n$ and $T_n \supseteq T'_n$ are sets with $|S_n| \sim |S'_n|$ and $|T_n| \sim |T'_n|$ as $n \to \infty$. If there exist bijections ϕ_n from S'_n to T'_n, then call ϕ_n an *asymptotic bijection* from S_n to T_n. When we have an asymptotic bijection, $|S_n| \sim |T_n|$. This idea can be used to obtain equalities as well as asymptotic results. We prove Dobinski's formula [1] for the number of partitions of an r-set,

$$B_r = e^{-1} \sum_{k=0}^{\infty} \frac{k^r}{k!}, \tag{1}$$

and the formula for the number of involutions on an r-set,

$$I_r = e^{-1/2} r! \sum_{k=0}^{\infty} \frac{\binom{2k}{r}}{k!\, 2^k}. \tag{2}$$

* Present address: Department of Mathematical Sciences, Drexel University, Philadelphia, Pa. 19104.

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One usually derives these from the exponential generating functions e^{e^x-1} [1, (5.4d)] and $e^{x+x^2/2}=e^{-1/2}e^{(x+1)^2/2}$ [4, Prob. 4.17d].

2. Set Partitions

Let $[n]=\{1,2,...,n\}$ and $[0]=\varnothing$, the empty set. Let $\mathscr{P}_k(n)$ be the collection of ordered samples without replacement of size k from $[n]$, $\mathscr{B}(r)$ be the set of partitions of $[r]$, and B^A be the set of maps from A to B. Note that $\mathscr{P}_0(n)=\{\varnothing\}$. Let

$$S_n=S'_n=\bigcup_{k=0}^{n}\mathscr{P}_{n-k}(n)\times[k]^{[r]}\qquad\text{and}\qquad T_n=\mathscr{B}(r)\times\bigcup_{k=0}^{n}\mathscr{P}_k(n).$$

Suppose that $\sigma\in\mathscr{P}_{n-k}(n)$ and $f\in[k]^{[r]}$. Let g be the unique order preserving bijection from $[k]$ to $[n]-\sigma$. If $\beta\in\mathscr{B}(r)$, order the blocks of β by their smallest elements and call the ith block β_i. Define

$$\phi_n(\sigma,f)=(\beta,(\sigma_1,...,\sigma_{n-k},\tau_1,...,\tau_j)),$$

where the blocks of β are the constant sets of f, $\tau_i=g(f(\beta_i))$ and j is the number of blocks of β. For example, if $r=5$,

$$\phi_7(4721,22112)=(\{125,34\},472153).$$

Let T'_n consist of those (β,π) where the ordered sample π has at least as many elements as β has blocks. Clearly ϕ_n is a bijection from S'_n to T'_n. To verify that it is an asymptotic bijection, note that $|T_n|>n!\,B_r$ and

$$|T_n-T'_n|\leqslant B_r\sum_{j<r}n(n-1)\cdots(n-j+1)=o(n!\,B_r).$$

Now (1) follows easily by noting that

$$|S_n|=n!\sum_{k=0}^{n}\frac{k^r}{k!}\qquad\text{and}\qquad|T_n|=n!\,B_r\sum_{k=0}^{n}\frac{1}{(n-k)!}.$$

We can prove $|S_n|=|T'_n|$ algebraically by a method like that in [3].

3. Involutions

Let $\mathscr{I}(r)$ be the set of involutions on $[r]$. Set

$$S_n=S'_n=\bigcup_{k=0}^{n}\mathscr{P}_r(2n-2k)\times(\mathscr{P}_k(n)\times[2]^{[k]})$$

and

$$T_n = \mathscr{I}(r) \times \bigcup_{k=0}^{n} (\mathscr{P}_k(n) \times [2]^{[k]}).$$

We regard an element of $\mathscr{P}_k(n) \times [2]^{[k]}$ as a signed, ordered k-sample from $[n]$. Suppose $\sigma \in \mathscr{P}_k(n)$ and $\tau \in \mathscr{P}_r(2n-2k)$. Let $\pm\sigma \in \mathscr{P}_k(n) \times [2]^{[k]}$ be σ with signs introduced. We regard r as an ordered r-sample from $\{x\colon |x| \in [n] - \sigma\}$. Suppose $\tau_{a_j} = -\tau_{b_j}$ with $a_j < b_j$ and τ_{c_j} are those τ_i's whose negatives are not in τ. Let

$$\phi_n(\tau, \pm\sigma) = ((a_1, b_1)(a_2, b_2)..., (\pm\sigma, \tau_{a_1}, \tau_{a_2},..., \tau_{c_1}, \tau_{c_2},...)).$$

For example, with $r = 6$ and overline indicating minus,

$$\phi_8(\bar{6}2\bar{3}35\bar{2}, 1\bar{8}) = ((26)(34), 1\bar{8}2\overline{36}5).$$

Let T'_n consist of those pairs of involutions and signed, ordered k-samples for which k is at least equal to the number of cycles in the involution (including fixed points as 1-cycles). Clearly ϕ_n is a bijection from S_n to T'_n. As in the previous section, $|T'_n| \sim |T_n|$. Now (2) follows easily by noting that

$$|S_n| = n!\, 2^n r! \sum_{k=0}^{n} \frac{\binom{2n-2k}{r}}{(n-k)!\, 2^{n-k}} \qquad \text{and} \qquad |T_n| = n!\, 2^n I_r \sum_{k=0}^{n} \frac{1}{(n-k)!\, 2^{n-k}}.$$

REFERENCES

1. L. COMTET, "Advanced Combinatorics: The Art of Finite and Infinite Expansions," Reidel, Dordrecht, 1974.
2. G. DOBINSKI, Summirungder Reihe $\sum n^m/n!$ für $m = 1, 2, 3, 4, 5,...$, *Grunert's Arch.* **61** (1877), 333–336.
3. J. R. GOLDMAN, An identity for fixed points of permutations, *Aequationes Math.* **13** (1975), 155–156.
4. J. RIORDAN, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.