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A COMBINATORIAL APPROACH TO MATRIX ALGEBRA

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"The theory of correspondence reaches far deeper than that of mere numerical congruity with which it is associated as the substance with the shadow"

James Joseph Sylvester

Introduction

To most contemporary mathematicians matrices and linear transformations are practically interchangeable notions. Indeed, the mainstream 'Bourbakian' establishment, with its profound disdain of the concrete, goes as far as to frown at the mere mention of the word 'matrix'.

To me, however, (as well as to a growing number of mathematical dissidents called 'combinatorialists') a matrix has nothing whatsoever to do with that intimidating abstract concept called 'a linear transformation between linear vector spaces". Instead, an $n \times n$ matrix is the 'blueprint' of all the possible edges one can draw on n given vertices, a determinant is the 'weight' of all permutation graphs and matrix-products represent paths (details later).

The purpose of this paper is to give a survey of this combinatorial interpretation of matrix algebra and to present elegant and illuminating proofs of five classical matrix identities.

In 1965, Dominque Foata [4, 2] gave a beautiful combinatorial proof of the celebrated MacMahon master theorem, thus setting the stage for combinatorial matrix algebra. Recently, two other elegant proofs have appeared: Straubing's proof of Cayley–Hamilton [9], and Orlin [8], Garsia [6] and Temperley [10] independently found a combinatorial proof of the matrix tree theorem.

I am going to present here new renditions of these three pearls, making them purely bijective and as succinct as possible. To them I am going to add two rubies of my own: a proof of $\det(AB) = (\det A)(\det B)$ and a new combinatorial proof (quite shorter than Foata's [5]) of Jacobi's $\det(e^A) = e^{\operatorname{tr} A}$.

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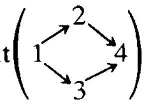
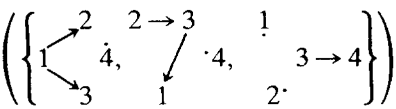
1. The set-up

For us, the entries of matrices $A=(a_{ij})$ are *not* numbers but rather com
indeterminates. We have n labeled vertices $\{1,\ldots,n\}$ and the *weight* of th
$i\to j$ is a_{ij}. A (directed) graph is a collection of edges and the *weight* of a g
the product of the weights of its edges. For example,

weight  $= a_{12}a_{13}a_{24}a_{34}.$

Whenever we have a set of objects possessing weights, we define the *weight*
set to be the sum of all the individual weights. For example

weight  $= a_{12}a_{13}+a_{23}a_{31}+a_{34}.$

A cycle is a directed graph whose edges are $i_1\to i_2, i_2\to i_3,\ldots, i_k\to$
some subset of the vertices $\{i_1,\ldots,i_k\}$. The weight of a cycle is $-a_{i_1i_2}a_{i_2i_3}\cdots$
(that is the negative of its weight *qua* graph). The weight of a disjoint uni
cycles is defined as the product of the weights of all constituent cycle
particular, it is readily seen that the weight of a permutation graph, whose
are $i\to\pi(i)$ $(i=1,\ldots,n)$ for some permutation π is equal to

$$(-1)^{\#\text{cycles}}\prod_{i=1}^{n} a_{i_{\pi(i)}}=(\operatorname{sgn}\pi)\prod_{i=1}^{n}(-a_{i_{\pi(i)}}).$$

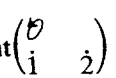
(This is so since the sign of an even cycle is -1 and the sign of an odd cycle
thus the sign of π is $(-1)^{\#\text{of even cycles}}$, taking $(-a_{ij})$ rather than (a_{ij}) gives
credit' to each odd cycle, making the total contribution to the left hand side
$(-1)^{\#\text{cycles}}$, as it should.)

We have thus obtained the following combinatorial interpretation o
determinant;

$$\det(-a_{ij})=\text{weight}(\mathscr{P}er(n)),$$

where $\mathscr{P}er(n)$ is the set of permutation graphs on the n vertices $\{1,\ldots$
Similarly, the principal minors of $(-a_{ij})$ corresponding to any subset of verti
the weight of the set of disjoint unions of cycles covering these vertices.
$\det(\delta_{ij}-a_{ij})$ (where δ_{ij} is the identity matrix) is the weight of the set of all dire
graphs that consist of disjoint union of cycles. For example, if $n=2$

$$\det(\delta_{ij}-a_{ij})=\begin{vmatrix}1-a_{11} & -a_{12}\\ -a_{21} & 1-a_{22}\end{vmatrix}$$

$$=\text{weight}\left(\dot{1}\quad\dot{2}\right)+\text{weight}$$ 

$$+\text{weight}\begin{pmatrix} & \circlearrowleft \\ \dot{1} & \dot{2}\end{pmatrix}+\text{weight}\begin{pmatrix}\circlearrowleft & \circlearrowleft \\ \dot{1} & \dot{2}\end{pmatrix}$$

$$+\text{weight}\begin{pmatrix}\dot{1} \rightleftarrows \dot{2}\end{pmatrix}.$$

If $A=(a_{ij})$ describes one kind of edges (called A-edges) and $B=(b_{ij})$ describes another kind of edges (called B-edges) then, for every pair (i,j), the (i,j) component of AB is the weight of the set of paths of length 2 from i to j such that the first edge is an A-edge and the second edge is a B-edge. This follows immediately from the definition of matrix multiplication. In particular the (i,j) entry of A^k is the weight of the set of paths of length k from i to j, where, of course,

$$\text{weight}(i \to i_2 \to i_3 \to \cdots \to i_k \to j) = a_{ii_2}a_{i_2i_3}\cdots a_{i_kj}.$$

2. Foata's proof of the MacMahon master theorem [2,4]

Let $A(m_1,\ldots,m_n)=$ coefficient of $x_1^{m_1}\cdots x_n^{m_n}$ in $(a_{11}x_1+\cdots+a_{1n}x_n)^{m_1}\cdots(a_{n1}x_1+\cdots+a_{nn}x_n)^{m_n}$. The MacMahon master theorem says that

$$\left(\sum A(m_1,\ldots,m_n)x_1^{m_1}\cdots x_n^{m_n}\right)\det(\delta_{ij}-a_{ij}x_j)=1. \tag{2}$$

Consider the collection $\mathscr{A}$ of all *pairs* (G,H) such that

(I) G is a directed graph, multiple edges and loops allowed such that

(i) For every vertex i, the number of outgoing edges equals the number of incoming edges,

(ii) for every vertex i, its outgoing edges are ordered from top to bottom (what computer folks would call a stack);

(II) H is a disjoint union of cycles (not necessarily covering all vertices).

For example, the following (G,H) is such a pair:

$$\begin{array}{lll}
G\text{: out of 1:} & 1\to 3 & H\text{: (13) (i.e., } 1\to 3\to 1)\\
 & 1\to 2 & \\
 & 1\to 1 & \\
 & 1\to 1 & \\
\text{out of 2:} & 2\to 3 & \\
 & 2\to 1 & \\
 & 2\to 2 & \\
\text{out of 3:} & 3\to 3 & \\
 & 3\to 3 & \\
 & 3\to 1 & \\
 & 3\to 2 &
\end{array}$$

The weight of an edge $i \to j$ is $a_{ij}x_j$ and let

$$\text{weight}(G, H) = (-1)^{\#\text{ cycles of } H} \cdot \text{product of all edge-weights of } G$$

For example, for the above (G, H)

$$\begin{aligned}\text{weight}(G, H) &= (-1)(a_{13}x_3)(a_{12}x_2)(a_{11}x_1)(a_{11}x_1)\\ &\quad\cdot (a_{23}x_3)(a_{21}x_1)(a_{22}x_2)\\ &\quad\cdot (a_{33}x_3)(a_{33}x_3)(a_{31}x_1)(a_{32}x_2)\\ &\quad\cdot (a_{13}x_3)(a_{31}x_1)\\ &= -a_{11}^2 a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32} a_{33}^2 x_1^5 x_2^3 x_3^5.\end{aligned}$$

We will prove (2) by showing that both the sides of (2) are equal to th thing, namely to

$$\text{weight}(\mathscr{A}) \stackrel{\text{def}}{=} \sum \text{weight}(G, H).$$

Let $\mathscr{G}$ be the set of all directed graphs satisfying (I) and $\mathscr{H}$ the set of all graphs satisfying (II). Clearly $\mathscr{A} = G \times \mathscr{H}$ and

$$\text{weight}(\mathscr{A}) = \text{weight}(\mathscr{G}) \cdot \text{weight}(\mathscr{H}).$$

By the remarks in Section 1,

$$\text{weight}(\mathscr{H}) = \det(\delta_{ij} - a_{ij}x_j).$$

In order to show that the left-hand side of (2) is equal to weight $(\mathscr{A})$ we wi to prove that

$$\text{weight}(\mathscr{G}) = \sum A(m_1, \ldots, m_n) x_1^{m_1} \cdots x_n^{m_n}.$$

Indeed, for every $(m_1, \ldots, m_n)$ consider the subset of $\mathscr{G}$ consisting of such that: for $i = 1, \ldots, n$, i has m_i outgoing edges (and therefore m_i in edges). Now for every i, you have m_i choices of choosing its outgoing ed the total weight of each choice is $(a_{i1}x_1 + \cdots + a_{in}x_n)$, implying that th weight of all m_i choices outgoing edges of i is $(a_{i1}x_1 + \cdots + a_{in}x_n)^{m_i}$. Do same thing for every single vertex shows that the weight of the set of having (for $i = 1, \ldots, n$) m_i edges out of i is $(a_{11}x_1 + \cdots + a_{1n}x_n)^{m_1} \cdots (a_n$ $\cdots + a_{nn}x_n)^{m_n}$. But we also have to take care of the fact that there are exactl edges coming into i $(i = 1, \ldots, n)$ and therefore weight$(\mathscr{G})$ = the $x_1^{m_1} \cdots x_n^{m_n}$ in the above product = $A(m_1, \ldots, m_n) x_1^{m_1} \cdots x_n^{m_n}$. Summing over $(m_1, \ldots, m_n)$ yields (5), which together with (4) and (3) yields

$$\text{weight}(\mathscr{A}) = \text{left-hand side of (2)}.$$

We will now prove that weight$(\mathscr{A}) = 1$, and thus complete the proof. Let's defin mapping from $\mathscr{A}$ to $\mathscr{A}$ as follows.

Given a pair (G, H), start at vertex 1 and walk along G in such a way that you always choose the top edge. Keep walking until either

Case I. You have encountered a previously visited vertex of G, or

Case II. You have come across a vertex of H.

In Case I we have transversed a complete cycle of G that is completely disjoint to the vertices of H. We remove this cycle from G and put it in H.

In Case II, we take the cycle of H to which that vertex belongs and move it from H to G. Also, we do it in such a way that these newcomer edges of G are placed on the top of the old edges.

For example, if

$$\begin{array}{ll} G = 1 \to 2 & H = \text{empty} \\ \quad\ 1 \to 1 & \\ \quad\ 2 \to 3 & \\ \quad\ 2 \to 1 & \\ \quad\ 3 \to 2 & \\ \quad\ 3 \to 3 & \end{array}$$

then the walk on G is $1 \to 2 \to 3 \to 2$, and Case I holds; thus the new (G, H), call it (G', H'), is

$$\begin{array}{ll} G' = 1 \to 2 & H' = (2, 3) \\ \quad\ \ 1 \to 1 & \\ \quad\ \ 2 \to 1 & \\ \quad\ \ 3 \to 3 & \end{array}$$

Now let's apply the mapping to (G', H'). The walk is $1 \to 2$, since vertex 2 is an H' vertex, belonging to $(2, 3)$. We remove $(2, 3)$ from H' and put its edges $2 \to 3$ and $3 \to 2$ in G' in their respective places on the top of the outgoing edges of 2 and 3 respectively. We get (G, H) back. Of course this is no coincidence, and it is readily seen that applying the mapping twice on any pair (G, H) reproduces it. In short, our mapping is an involution and therefore, of course, a bijection. Since there is 'conservation of edges' in (G, H) the absolute value of the weight remains the same, but since the parity of the number of cycles of H changes, the sign changes. Thus all the terms of $\text{weight}(\mathscr{A}) = \sum \text{weight}(G, H)$ can be arranged in mutually cancelling pairs, except to the only element of $\mathscr{A}$ on which the involution cannot be defined, namely the 'trivial' pair (empty, empty) whose weight is 1. Thus

$$\text{weight}(\mathscr{A}) = 1 = \text{right-hand side of (2)}.$$

This completes the proof.

3. Straubing's proof of the Cayley–Hamilton theorem [9]

Let A be an $n\times n$ matrix and let $P(\lambda)=\det(\lambda I-A)$ then the Cayley–Ham theorem says that the $n\times n$ matrix $P(A)$ is the zero matrix. Spelled out in f says that

$$\begin{aligned}&A^n+(-a_{11}-a_{22}-\cdots-a_{nn})A^{n-1}\\&\quad+(\text{sum of all } 2\times 2 \text{ principal minors, of } -A)A^{n-2}\\&\quad+\cdots+(\text{sum of all } k\times k \text{ principal minors of } -A)A^{n-k}\\&\quad+\cdots+\det(-A)\equiv 0.\end{aligned}$$

We have to prove that every entry of the matrix on the left hand side of (*) equal to zero.

Fix i and j and let $\mathscr{A}=\mathscr{A}(i,j)$ be the set of pairs (P,C) such that

(i) P is a path from i to j,

(ii) C is a disjoint union of cycles,

(iii) The total number of edges of P and C combined equals n.

The weight of an edge $k\to m$ is a_{km} and

$$\text{weight}(P,C)=(-1)^{\#\text{ cycles of } C}[\text{product of all edge-weights of } C \text{ and } P]$$

For example, if $i=1, j=2, n=5$, $(1\to 3\to 2,\ (1)(3,5))$ is an element of whose weight is $(-1)^2(a_{13}a_{32})[(a_{11})(a_{35}a_{53})]$.

Now we claim that

$$\text{weight}(\mathscr{A}(i,j))=(i,j) \text{ entry of the left-hand side of } (*).$$

Indeed, the path P may be of any length $n-k$ for $0\leqslant k\leqslant n$. The weight of th of paths of length $n-k$ from i to j is exactly the (i,j) entry of A^{n-k}. Now have k edges left to form disjoint cycles, and you have the freedom to choose a k-element subset of $\{1,\ldots,n\}$ for your vertices. The weight of the set of all th is (by the remarks of Section 1) equal to the sum of all $k\times k$ principal minors $-A$. Summing over all $0\leqslant k\leqslant n$ gives (7).

The proof will be completed once we show that for every i,j

$$\text{weight}(\mathscr{A}(i,j))=0.$$

To this end we will introduce the following mapping from $\mathscr{A}(i,j)$ to itself. Give (P,C) start at i and walk along the path P until you either

Case I. Come to a previously visited vertex of P, or

Case II. come to a vertex that belongs to one of the cycles of C.

In Case I you have transversed a cycle of P whose vertices are disjoint to all th cycles of C. You remove that cycle from P and join it to C.

In Case II you remove that cycle from C and insert it (at that vertex) in P.

Example. $n=5,\ i=1,\ j=3$

$$(1\to 2\to 3\to 2\to 3;(5))\leftrightarrow(1\to 2\to 3;(23),(5))$$
$$(1\to 3\to 3;\ (3,4,5))\leftrightarrow(1\to 3\to 4\to 5\to 3\to 3;\emptyset).$$

It is readily seen that this mapping is an involution defined on every element (P, C) of $\mathscr{A}$. (Let the number of vertices (= number of edges) of C be k, and suppose that the vertices of P are disjoint from those of C. Then P has as many vertices as edges ($n-k$ of them) and therefore must contain a cycle.) By 'conservation of edges' the absolute value of the weight stays the same, but since the parity of the number of cycles of C changes, the sign of the weight is reversed. Thus, all the elements of weight($\mathscr{A}$) can be arranged in mutually cancelling pairs and their sum is therefore zero.

4. A combinatorial proof of the matrix tree theorem [6, 8, 10]

Consider directed graphs on the n vertices $\{1,\ldots,n\}$. A tree rooted at n is a directed graph without cycles such that every vertex has exactly one outgoing edge except to the root n that has no outgoing edges. Let $\mathscr{T}=\mathscr{T}(n)$ be the set of trees rooted at n. The weight of an edge $k\to m$ is a_{km} and the weight of a tree (or for that matter any directed graph) is the product of the edge-weights.

The matrix-tree theorem says that weight($\mathscr{T}(n)$) equals the determinant

$$\begin{vmatrix} a_{12}+\cdots+a_{1n} & -a_{12} & \cdots & -a_{1,n-1} \\ -a_{21} & a_{21}+\cdots+a_{2n} & & -a_{2,n-1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & a_{n-1,1}+\cdots+a_{n-1,n} \end{vmatrix}. \tag{9}$$

Let $\mathscr{B}$ be the set of pairs (B, C) such that

(i) B is a directed graph such that for a certain subset V_B of $\{1,\ldots,n-1\}$ there is exactly one edge going out of every vertex of V_B. The end vertex of each edge may be any vertex of $\{1,\ldots,n\}$ *except* its origin (i.e., no slings allowed);

(ii) C is a collection of disjoint cycles, of length $\geqslant 2$, on the set of vertices V_C, V_C being the complement of V_B with respect to $\{1,\ldots,n-1\}$.

The weight of a pair (B, C) is defined by

$$\text{weight}(B, C)=(-1)^{\#\text{ cycles of } C}[\text{product of all edge-weights of } B \text{ and } C].$$

For example ($n=5$)

$$\text{weight}(1\to 5, 3\to 5;\ (2,4))=(-1)^1 a_{15}a_{35}a_{24}a_{42}.$$

It is readily seen that (9) = weight($\mathscr{B}$).

Define the following mapping on $\mathscr{B}$. Given (B, C) look at *all* cycles, both of B (if any) and of C. Pick the cycle that contains the lowest vertex and change affiliation (if it belonged to B put it in C and vice versa). For example ($n = 6$)

$$(1 \to 2, 2 \to 1, 4 \to 6; (35)) \leftrightarrow (4 \to 6; (12)(35))$$

$$(1 \to 6, 2 \to 6; (345)) \leftrightarrow (1 \to 6, 2 \to 6, 3 \to 4, 4 \to 5, 5 \to 3; \emptyset).$$

It is not hard to see that we have a sign reversing involution that is defined on all elements of $\mathscr{B}$ that have cycles. The only survivors are those elements of $\mathscr{B}$ of the form $(B, \emptyset)$ where B has no cycles, i.e., is a tree! Thus weight($\mathscr{A}$) = weight($\mathscr{T}$) and this completes the proof that weight($\mathscr{T}$) equals (9).

5. det(AB) = (det A)(det B)

The matrix AB represents compound edges $i \xrightarrow{A} k \xrightarrow{B} j$ with weight $a_{ik}b_{kj}$, where k can be any vertex. Let $\text{weight}_A(\pi) = (\text{sgn}\,\pi)a_{1\pi(1)} \cdots a_{n\pi(n)}$, $\text{weight}_B(\pi) = (\text{sgn}\,\pi)b_{1\pi(1)} \cdots b_{n\pi(n)}$. Let $\mathscr{P}er(n)$ denote the set of permutations on $\{1, \ldots, n\}$, then $\det A = \text{weight}_A(\mathscr{P}er(n))$, $\det B = \text{weight}_B(\mathscr{P}er(n))$. What is $\det(AB)$?

Let $Z(n)$ be the set of pairs (f, π) where f is *any* mapping $\{1, \ldots, n\} \to \{1, \ldots, n\}$ and π is a permutation. Let

$$\text{weight}(f, \pi) = (\text{sgn}\,\pi)(a_{1f(1)}b_{f(1)\pi(1)}) \cdots (a_{if(i)}b_{f(i)\pi(i)}) \cdots (a_{nf(n)}b_{f(n)\pi(n)}).$$

A moment's reflection would convince you that

$$\det(AB) = \text{weight}(Z(n)).$$

An element of $Z(n)$ is a *good guy* if f is a permutation. Then of course $f^{-1} \circ \pi$ is a permutation and $\text{weight}(f, \pi) = \text{weight}_A(f)\text{weight}_B(f^{-1} \circ \pi)$. Thus

$$\sum_{(f,\pi)\ \text{good}} \text{weight}(f, \pi) = (\det A)(\det B). \tag{10}$$

In order to prove that $\det(AB)$, which we said was equal to weight($Z(n)$), is equal to $(\det A)(\det B)$, we have to show only, thanks to (10), that

$$\sum_{(f,\pi)\ \text{bad}} \text{weight}(f, \pi) = 0. \tag{11}$$

Once again we have to find a killer involution. If (f, π) is a bad guy, all it means is that f is *not* a permutation, i.e, there exist b, i and i' such that $f(i) = b$ and $f(i') = b$, or in a more picturesque notation there exist A-edges $i \xrightarrow{A} b$ and $i' \xrightarrow{A} b$. Pick the smallest such b, and for that b, the smallest such i and i'.

Case I: i and i' belong to the *same* cycle of π. The cycle to which both i and i' belong looks as follows:

$$i \xrightarrow{A} b \xrightarrow{B} \pi(i) \xrightarrow{A} \text{whatever} \cdots \xrightarrow{B} i' \xrightarrow{A} b \xrightarrow{B} \pi(i') \to \text{blablabla} \cdots \to i.$$

What you have to do is break this long cycle into two cycles:

$$i \xrightarrow{A} b \xrightarrow{B} \pi(i') \to \text{blablabla} \to i$$

and

$$\pi(i) \xrightarrow{A} \text{whatever} \xrightarrow{B} i' \xrightarrow{A} b \xrightarrow{B} \pi(i).$$

(Note that the underlying permutation changed from π to π times the transposition (i, i').)

Case II: i and i' belong to *different* cycles of π. Let these cycles be

$$i \xrightarrow{A} b \xrightarrow{B} \pi(i) \xrightarrow{A} \text{blablabla} \cdots \xrightarrow{B} i$$

and

$$i' \xrightarrow{A} b \xrightarrow{B} \pi(i') \xrightarrow{A} \text{whatever} \cdots \xrightarrow{B} i'.$$

In this case what you have to do is to combine them into one cycle:

$$i \xrightarrow{A} b \xrightarrow{B} \pi(i') \xrightarrow{A} \text{whatever} \cdots \xrightarrow{B} i' \xrightarrow{A} b \xrightarrow{B} \pi(i) \rightarrow \text{blablabla} \cdots \rightarrow i.$$

(Note that the underlying permutation changed from π to π times the transposition (i, i').)

Example. $n = 6$

$$(1 \xrightarrow{A} 4 \xrightarrow{B} 2 \xrightarrow{A} 2 \xrightarrow{B} 5 \xrightarrow{A} 3 \xrightarrow{B} 6 \xrightarrow{A} 3 \xrightarrow{B} 3 \xrightarrow{A} 2 \xrightarrow{B} 4 \xrightarrow{A} 2 \xrightarrow{B} 1)$$

$$\Leftrightarrow$$

$$(1 \xrightarrow{A} 4 \xrightarrow{B} 2 \xrightarrow{A} 2 \xrightarrow{B} 4 \xrightarrow{A} 2 \xrightarrow{B} 1)(2 \xrightarrow{B} 5 \xrightarrow{A} 3 \xrightarrow{B} 6 \xrightarrow{A} 3 \xrightarrow{B} 3 \xrightarrow{A} 2).$$

It is readily seen that what we have here is a sign reversing involution defined on all the bad guys and thus the sum of the weights of all the bad guys is 0. This proves (11) which together with (10) completes the proof of $\det(AB) = (\det A)(\det B)$.

6. A new combinatorial proof of Jacobi's $\det(e^A) = e^{\operatorname{tr} A}$

The first to realize that Jacobi's identity has anything to do with combinatorics was Jackson [7] who gave it a combinatorial *interpretation*. Foata [5] then went on to give an elegant combinatorial proof. We are now going to give another combinatorial proof that is shorter and more direct.

$e^A = \sum A^k/k!$ is the *exponential* generating function of paths of all length. Namely, writing $B = e^A$, $B = (b_{ij})$ we have

$$\frac{1}{k!}\text{weight[set of all paths from } i \text{ to } j \text{ of length } k]$$

$$= \text{the sum of all terms in } b_{ij} \text{ of total degree } k.$$

Now for $m = 0, 1, 2, \ldots,$ let $\mathcal{B}_m$ be the set of objects of the form $(\pi, P_{i\pi(i)},$

$i=1,\ldots,n)$ where

(1) π is a permutation of $\{1,\ldots,n\}$;

(2) For $i=1,\ldots,n$, $P_{i\pi(i)}$ is a path from i to $\pi(i)$;

(3) The total number of edges of all paths is m;

(4) The *edges* are labeled by distinct labels from $\{1,\ldots,m\}$ in such a way tha they are increasing along every path.

The weight of such an object is sgn π times the product of all edge-weights, th product of an edge $k\to l$ being a_{kl}.

For example if $n=4$, $m=15$

$$\pi = 1\longrightarrow 2, 2\longrightarrow 1, 3\longrightarrow 3, 4\longrightarrow 4$$

$$P_{12}\colon 1\xrightarrow{1}2\xrightarrow{3}2\xrightarrow{5}2$$

$$P_{21}\colon 2\xrightarrow{4}1\xrightarrow{6}1\xrightarrow{9}4\xrightarrow{11}1$$

$$P_{33}\colon 3\xrightarrow{2}4\xrightarrow{8}1\xrightarrow{10}2\xrightarrow{14}3$$

$$P_{44}\colon 4\xrightarrow{7}4\xrightarrow{12}1\xrightarrow{13}4\xrightarrow{15}4$$

is one member of $\mathscr{B}_{15}$ whose weight is

$$(+1)(a_{12}a_{22}a_{22})(a_{21}a_{11}a_{14}a_{41})(a_{34}a_{41}a_{12}a_{23})(a_{44}a_{41}a_{14}a_{44}).$$

By general properties of exponential generating functions we have

$$\det(e^A)=\sum_{m=0}^{\infty}\frac{1}{m!}\text{weight}(\mathscr{B}_m).$$

We are now going to define an involution $\mathscr{B}_m\to\mathscr{B}_m$ (for every m) that is going get rid of most of the terms in weight$(\mathscr{B}_m)$.

Let $j\xrightarrow{s}i$ be the edge of highest label s for which $j\neq i$. This edge mı necessarily belong to $P_{\pi^{-1}(i)i}$ which has the form

– first path:

$$P_{\pi^{-1}(i)i}\colon \pi^{-1}(i)\longrightarrow\text{whatever}\cdots\longrightarrow j\xrightarrow{s}i\xrightarrow{s_1}i\xrightarrow{s_2}i\longrightarrow\cdots\xrightarrow{s}$$

where $s=s_0$ and $r\geqslant 0$.

Now consider $P_{\pi^{-1}(j)j}$

– second path:

$$P_{\pi^{-1}(j)j}\colon \pi^{-1}(j)\longrightarrow\text{blablabla}\cdots\xrightarrow{t_0}j\xrightarrow{t_1}j\xrightarrow{t_2}\cdots\xrightarrow{t_l}j\quad(l\geqslant 0).$$

Let $0\leqslant\alpha\leqslant l$ be the only α such that $t_\alpha<s<t_{\alpha+1}$. The involution consists swapping the portion $\xrightarrow{s}i\xrightarrow{s_1}i\longrightarrow\cdots\xrightarrow{s_r}i$ of the first path and (the possi

empty) portion $\xrightarrow{t_{\alpha+1}} j \xrightarrow{t_{\alpha+2}} j \longrightarrow \cdots \xrightarrow{t_l} j$ of the second path, getting for the transformed object

- first path:

$$P_{\pi^{-1}(i)j}\colon \pi^{-1}(i) \longrightarrow \text{whatever} \cdots \longrightarrow j \xrightarrow{t_{\alpha+1}} j \longrightarrow \cdots \xrightarrow{t_l} j$$

- second path:

$$P_{\pi^{-1}(j)i}\colon \pi^{-1}(j) \longrightarrow \text{blablabla} \cdots \xrightarrow{t_\alpha} j \xrightarrow{s} i \xrightarrow{s_1} i \longrightarrow \cdots \xrightarrow{s_r} i$$

Example. $n = 3$

$$\left.\begin{array}{l} \pi = \begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix} \\ P_{12}\colon 1 \xrightarrow{1} 2 \xrightarrow{3} 2 \xrightarrow{6} 2 \xrightarrow{13} 2 \\ P_{23}\colon 2 \xrightarrow{2} 1 \xrightarrow{7} 3 \xrightarrow{8} 3 \xrightarrow{9} 3 \xrightarrow{10} 3 \\ P_{31}\colon 3 \xrightarrow{4} 1 \xrightarrow{5} 1 \xrightarrow{11} 1 \xrightarrow{12} 1 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \pi = \begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \\ P_{12}\colon 1 \xrightarrow{1} 2 \xrightarrow{3} 2 \xrightarrow{6} 2 \xrightarrow{13} 2 \\ P_{21}\colon 2 \xrightarrow{2} 1 \xrightarrow{11} 1 \xrightarrow{12} 1 \\ P_{33}\colon 3 \xrightarrow{4} 1 \xrightarrow{5} 1 \xrightarrow{7} 3 \xrightarrow{8} 3 \xrightarrow{9} 3 \xrightarrow{10} 3. \end{array}\right.$$

Since we have 'conservation of edges', the absolute value of the weight is retained, but the sign changes, since the underlying permutation has been multiplied by the transposition (i, j). Thus the involution gets rid of all the terms in weight($\mathscr{B}_m$) corresponding to elements of $\mathscr{B}_m$ on which the involution makes sense. The only survivors of weight($\mathscr{B}_m$) are weights of elements on which the involution cannot be defined, call these elements $\mathscr{C}_m$. Thus weight($\mathscr{B}_m$) = weight($\mathscr{C}_m$), where $\mathscr{C}_m$ are those objects all of whose edges are slings, that is, edges from a vertex to itself $i \to i$. The underlying permutation for all the members of $\mathscr{C}_m$ must necessarily be the identity permutation and all paths have the form

$$\begin{array}{l} P_{11}\colon 1 \to 1 \to 1 \to \cdots \to 1 \\ \cdot \\ \cdot \\ \cdot \\ P_{nn}\colon n \to n \to \cdots \to n \end{array}$$

with appropriate edge labels. Since the exponential generating function for paths of the form $i \to i \to \cdots \to i$ is e^{a_i} it follows that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \text{weight}(\mathscr{C}_m) = e^{a_1} e^{a_2} \cdots e^{a_n} = e^{\operatorname{tr} A}.$$

Thus

$$\det(e^A) = \sum \frac{1}{m!} \text{weight}(\mathscr{B}_m) = \sum \frac{1}{m!} \text{weight}(\mathscr{C}_m) = e^{\operatorname{tr} A}$$

This completes the proof.

7. Exercises

The difficulty rating of these exercises follows the famous Knuth ranges from 0 (outright trivial) to 50 (impossible).

1. (10) Starting from the combinatorial definition of the determinant the determinant vanishes if two rows are identical.

2. (12) Give a combinatorial interpretation to an arbitrary minor (nc ily principal) obtained by choosing rows $i_1, \ldots, i_k$ and columns $j_1, \ldots$

3. (20) Using Exercise 2, prove the Laplace expansion of the deter

4. (39) Go through Aitken's book [1] and try to prove combinatorial results as possible.

5. (28) Using a proof similar to Straubing's, prove the following ide be *any* integer then if A is an $n \times n$ matrix $(\operatorname{tr} A^m)+(\text{sum of } 1\times$ pr minors of $-A)$ $(\operatorname{tr} A^{m-1})+\cdots+(\text{sum of } k\times k$ principal minor $(\operatorname{tr} A^{m+k})+\cdots+(\text{sum of } m\times m$ principal minors of $-A)$ $m \equiv 0$.

Note that for $m \geq n$ it is a trivial consequence of the Cayley–Hamilto the and that for A diagonal these are Newton's identitities.

6. (45) Another way of stating the matrix tree theorem is to say that the of the set of trees rooted at n equals the (n, n) minor of the determinant matrix (A_{ij}) defined by $A_{ii} = \sum_{j\neq i} a_{ij}$, $A_{ij} = -a_{ij}$ $(i \neq j)$. Find what is enumera an arbitrary (not necessarily principal) minor and prove the so-called "all matrix tree theorem". (For a proof see Chaiken's [3] interesting paper.)

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