Thus the number of n-tone rows is

$$\tfrac{1}{4}[(n-1)!+(n-1)(n-3)\cdots(2)] \qquad \text{if } n \text{ is odd;}$$
$$\tfrac{1}{4}[(n-1)!+(n-2)(n-4)\cdots(2)(1+n/2)] \qquad \text{if } n \text{ is even.}$$

For example, there are 9985920 twelve tone rows, a fact which does not seem to be in the literature.

Appendix. Computation of numbers of fixed elements.

1. $I(0,a_2,\dots,a_n)=(0,-a_2,\dots,-a_n)\sim(0,a_2,\dots,a_n)$ implies $a_i\equiv -a_i \pmod n$ for $i=2,\dots,n$ since no transposition is allowed because the first element 0 is fixed. There is at most one nonzero solution to $x\equiv -x \pmod n$, but that is not enough to fill out the permutation.

2. $R(a_1,\dots,a_n)=(a_n,\dots,a_1)\sim(a_1,\dots,a_n)$ implies that a t exists such that $a_1\equiv a_n+t$, $a_2\equiv a_{n-1}+t,\dots \pmod n$. If n is odd, the middle element is fixed and no transposition is allowed: $t=0$. But then $a_n=a_1$, a contradiction. If n is even, the first and last congruences imply that $2t\equiv 0$; hence, $t=0$ or $t=n/2$. The first is impossible just as when n is odd, but the other gives fixed permutations. Since $a_1=0$, $a_n=n/2$. For a_2, we can choose any of $n-2$ elements, and this determines a_{n-1}. For a_3, we have $n-4$ choices, etc.

3. $IR(a_1,\dots,a_n)=(-a_n,\dots,-a_1)\sim(a_1,\dots,a_n)$ implies that a t exists such that $a_1+a_n\equiv t$, $a_2+a_{n-1}\equiv t,\dots$. The last congruence for n odd is $2a_{(n+1)/2}\equiv t \pmod n$. Clearly it is not important that we fix the first element as 0; we could fix $a_{(n+1)/2}$ as 0 and obtain the same count. Thus we may assume that $t=0$, $a_{(n+1)/2}=0$. This allows $n-1$ choices for a_1, with a_n thus determined, $n-3$ choices for a_2, etc. For n even, we fix $a_1=0$, $a_n\neq t\neq 0$. A little thought shows that t must be odd in order for us to complete the permutation. If $t=2k$, then there is no mate for k in the permutation. There are thus $n/2$ choices for $t=a_n$. For a_2, there are $n-2$ choices, with a_{n-1} determined thereby, etc.

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BIJECTING EULER'S PARTITIONS-RECURRENCE

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A partition of an integer n is a nonincreasing sequence of positive integers $\lambda(1)\geqslant\lambda(2)\geqslant\cdots\geqslant\lambda(t)>0$, such that $\lambda(1)+\cdots+\lambda(t)=n$. The set of partitions of n is denoted $\mathrm{Par}(n)$ and its cardinality $|\mathrm{Par}(n)|$ is written $p(n)$. For example,

$$\mathrm{Par}(5)=\{5;4,1;3,2;3,1,1;2,2,1;2,1,1,1;1,1,1,1,1\} \quad\text{and}\quad p(5)=7.$$

There is no closed form formula for $p(n)$ but Euler ([1], p. 12) gave a very efficient way for compiling a table of $p(n)$ by proving the recurrence

$$\sum_{j\text{ even}} p(n-a(j))=\sum_{j\text{ odd}} p(n-a(j)),\quad \text{where } a(j)=(3j^2+j)/2. \tag{1}$$

Euler used generating functions to prove this formula. Garsia and Milne [2] gave a very nice bijective proof of (1), utilizing their Involution Principle. We are going to give another bijective proof which does not require any iterations and is very simple. Indeed,

$$\phi\colon \bigcup_{j \text{ even}} \mathrm{Par}(n - a(j)) \leftrightarrow \bigcup_{j \text{ odd}} \mathrm{Par}(n - a(j)),$$

given below does the job.

Let $(\lambda) = (\lambda(1), \dots, \lambda(t)) \in \mathrm{Par}(n - a(j))$. Then define ϕ by

$$\phi((\lambda)) = \begin{cases} (t + 3j - 1, \lambda(1) - 1, \dots, \lambda(t) - 1) \in \mathrm{Par}(n - a(j - 1)), & \text{if } t + 3j \geqslant \lambda(1), \\ (\lambda(2) + 1, \dots, \lambda(t) + 1, 1, 1, \dots, 1) \in \mathrm{Par}(n - a(j + 1)), & \text{if } t + 3j < \lambda(1), \\ \text{where there are } \lambda(1) - 3j - t - 1 \text{ 1's at the end.} \end{cases}$$

Note that applying ϕ twice yields the identity mapping, thus $\phi = \phi^{-1}$ and ϕ is invertible.

EXAMPLE. $n = 21$.

$$\phi(5,5,4,3,2) = (7,4,4,3,2,1).$$

Here $(5,5,4,3,2,) \in \mathrm{Par}(19) = \mathrm{Par}(n - a(1))$ so $j = 1$. The number of parts t, is 5 and we have $t + 3j \geqslant \lambda(1)$, since $5 + 3 \geqslant 5$. Now consider $\phi(7,4,4,3,2,1)$; here $j = 0$, $t = 6$, $\lambda(1) = 7$ and $6 + 0 < 7$. Also $\lambda(1) - 3j - t - 1 = 7 - 0 - 6 - 1 = 0$ so we do not add any 1's at the end and $\phi(7,4,4,3,2,1) = (5,5,4,3,2)$.

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THE TEACHING OF MATHEMATICS

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ON THE CONVOLUTION OF CAUCHY DISTRIBUTIONS

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The characteristic function of a probability distribution is usually too advanced a topic for a first undergraduate course in mathematical statistics and the more limited moment generating function is often used instead. In teaching the distribution of sums of independent random variables such as normal, gamma, or uniform, I supplement the use of the moment generating function with the convolution formula,

$$f * g(u) = \int_{-\infty}^{\infty} f(x)\, g(u - x)\, dx. \tag{1}$$

For sums of independent Cauchy random variables the moment generating function does not apply and the use of the convolution formula is difficult. Undoubtedly, it is generally understood that if f and g are Cauchy densities, a partial fraction decomposition of the integrand in (1) should lead to an explicit evaluation of the convolution integral but I do not find the details worked out anywhere. (See comment in Feller, [1], p. 51.) The purpose of this note is to outline