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Resurrecting the Asymptotics of Linear Recurrences*

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Once on the forefront of mathematical research in America, the asymptotics of the solutions of linear recurrence equations is now almost forgotten, especially by the people who need it most, namely combinatorists and computer scientists. Here we present this theory in a concise form and give a number of examples that should enable the practicing combinatorist and computer scientist to include this important technique in her (or his) asymptotics tool kit. © 1985 Academic Press, Inc.

INTRODUCTION

Enumerative combinatorics studies sequences $\{A_n\}$ that count families of sets defined by some combinatorial rule. For example, $\{F_n\}$, the Fibonacci numbers, count the number of rabbits after n generations; $\{2^n - 1\}$ counts the number of grains of wheat that the inventor of chess had after the nth square; $\{n!\}$ counts the number of permutations on n objects, etc.

Given any such sequence it is natural to inquire: "How fast does it grow as n gets bigger?" This is why asymptotics is so important in combinatorics.

Asymptotics is even more important in computer science where it is used to rate algorithms. For example, an algorithm is considered fast if it is $O(n)$ or $O(n \log n)$, n being the input size. Ironically, an algorithm is considered tractable if it is $O(n^{10^{10}})$ but intractable if it is $O(e^{10^{-10}n})$.

It should not come as a surprise, therefore, that both combinatorists [3] and computer scientists [10, 1.2.11] study asymptotic methods. What *is* surprising is that there seems to be a general lack of knowledge about a powerful asymptotic technique pioneered by Poincaré [13] and culminating in the works of George D. Birkhoff [4] and his student W. J. Trjitzinsky [5]. This method enables one to find the exact asymptotics of sequences defined by a large class of linear recurrences, up to a positive

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constant multiple. Since many combinatorial families are defined recursively and the corresponding sequence of cardinalities $a(n)$ satisfies a natural recurrence, the Birkhoff technique lends itself to many problems in combinatorial asymptotics.

This paper is a joint effort of a continuous mathematician and a discrete mathematician. Both of us were interested in recurrence equations, but from different points of view. D. Zeilberger has shown the great usefulness of Sister Celine's technique [19] for obtaining recurrence relations while J. Wimp has demonstrated that in many cases of great importance, the coefficients of the recurrence equation can be obtained directly in closed form [8, 9, 15, 16].

In Section 1 we will introduce several examples of combinatorial families that are counted by solving a recurrence equation. Then we will go on, in Section 2, to give an account of the Birkhoff–Trjitzinsky method and finally, in Section 3, we will give a few examples of combinatorial interest.

1. Combinatorial Families Counted by Solving a Linear Recurrence Equation

Example 1.1. The set of permutations on n objects, S_n, can be defined recursively by $S_{n+1} = [n+1] \times S_n$ and the cardinality $a(n) = |S_n|$ satisfies the obvious recurrence $a(n+1) = (n+1)\, a(n)$ (whose solution is called $n!$). The asymptotics for $n!$ is the famous Stirling formula that is usually derived by applying the Euler–Maclaurin formula [10, p. 120] to $\log n!$ and then exponentiating. We recommend that the readers consult Batchelder's book [2, p. 41] for a novel derivation of Stirling's formula using the Birkhoff technique.

Example 1.2. The Fibonacci numbers F_n count the number of rabbits after n generations [10, 1.2.8] or alternatively, the number of possible ways of climbing up n steps where one may either take one or two steps at a time. They satisfy the famous recurrence $F_n = F_{n-1} + F_{n-2}$. Since this is a constant coefficient recurrence, one can write down the solution exactly [10, 1.2.8] as

$$F_n = \frac{1}{2\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

and of course

$$F_n \sim \frac{1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

provides the complete asymptotics.

EXAMPLE 1.3. An involution is a permutation that is equal to its inverse or, equivalently, a permutation which is a product of disjoint 1-cycles and 2-cycles. Let $t(n)$ denote the number of involutions on $\{1,..., n\}$. Since "n" may belong to either a 1-cycle ($t(n-1)$ possibilities) or a 2-cycle ($n-1$ possible companions and $t(n-2)$ ways to arrange the remaining $n-2$ elements) the natural recurrence satisfied by $t(n)$ is $t(n)=t(n-1)+(n-1)\,t(n-2)$. The first to evaluate the asymptotics of $t(n)$ were Moser and Wyman [12], who used the exponential generating function $\sum (t(n)\,z^n/n!)=\exp(z+z^2/2)$. Knuth [11, p. 67] gave another derivation starting from $t(n)=\sum (n!/((n-2k)!\,2^k k!))$, approximating the binomial coefficients by Stirling's formula and then using the Euler–Maclaurin formula. In Section 2 we will derive the asymptotics by using the Birkhoff method.

EXAMPLE 1.4. Let $u_r(n)=u(n)$ be the number of permutations whose cycle structure consists of 1, 2,..., r-cycles but no cycle of length bigger than r. Both the Wyman–Moser method and the method described by Knuth become more complicated, since the former requires the generating function $\exp(z+z^2/2+z^3/3+\cdots+z^r/r)$ while the latter involves a multiple sum (with $r-1\ \sum s$). Using the Birkhoff technique applied to the natural recurrence

$$u(n)=u(n-1)+(n-1)\,u(n-2)+(n-1)(n-2)\,u(n-3)$$
$$+\cdots+(n-1)\cdots(n-r+1)\,u(n-r)$$

one can easily obtain the asymptotics. This is done in Section 3.

2. THE BIRKHOFF–TRJITZINSKY METHOD

The difference equation we shall consider will be written in the form

$$\sum_{\nu=0}^{\sigma} C_\nu(n)\,y(n+\nu)=0, \qquad C_0(n)=1, \qquad C_\sigma(n)\neq 0, \qquad n=0, 1, 2,.... \tag{1}$$

The integer $\sigma\geqslant 1$ is called the order of the equation. The restrictions on C_0, C_σ are sufficient to guarantee that, corresponding to arbitrary initial conditions of the form

$$y(k)=\alpha_k, \qquad k=0, 1, 2,..., \sigma-1, \tag{2}$$

the equation will possess a unique solution $y(n)$, $n=0, 1, 2,....$

In what follows we assume the coefficients in (1) possess representations as generalized Poincaré series

$$C_\nu(n) \sim n^{K_\nu/\omega}[c_{0\nu} + c_{1\nu}n^{-1/\omega} + c_{2\nu}n^{-2/\omega} + \cdots], \quad \nu = 1, 2, \ldots, \sigma, \quad n \to \infty, \tag{3}$$

where K_ν is an integer, ω is an integer $\geqslant 1$ independent of ν and $c_{0\nu} \neq 0$ unless $C_\nu(n) \equiv 0$. We assume (3) is written with the smallest possible value of ω. Note that any difference equation with rational coefficients is of the form (1), (3). Further, if one is dealing with the nonhomogeneous equation

$$\sum_{\nu=0}^{\sigma} C_\nu(n)\, y(n+\nu) = h(n), \qquad n = 0, 1, 2, \ldots, \tag{4}$$

and $h(n) \neq 0$ has a representation of the form (3), it is apparent that solutions of this equation may be simply related to solutions of a homogeneous equation of order one higher $(\sigma + 1)$. To obtain this equation, divide (4) by $h(n)$, then replace n by $n+1$ and subtract the two equations. The coefficients in the resulting equation will have the required asymptotic form. Thus the difference equation (1) is sufficiently general to include nearly all difference equations encountered in practice.

A set of ν functions $z^{(j)}(n)$ are called *linearly independent* if the determinant

$$|z^{(j+1)}(n+i)|, \qquad i, j = 0, 1, \ldots, \nu - 1,$$

does not vanish for any $n = 0, 1, 2, \ldots$. The classical theory of difference equations asserts that Eq. (1) possesses a set of σ linearly independent solutions which constitute a basis of the solution space. Such a set of solutions is called a *fundamental set*. Any solution, i.e., one satisfying (2), can be written as a linear combination of such a set.

The theory of Birkhoff and Trjitzinsky shows that there exists a fundamental set for (1) whose members share an unusual property: each has an asymptotic expansion which consists of an exponential leading term multiplied by a linear combination of descending series of the form (3) (where, however, ω may be replaced by an integral multiple of ω) multiplied by terms $(\ln n)^j$, $j = 0, 1, \ldots$.

In this section, we shall briefly describe the theory and discuss the construction of such series. We begin with several definitions.

Consider the series

$$e^{Q(\rho;n)} s(\rho; n), \tag{5}$$

$$Q(n) \equiv Q(\rho; n) := \mu_0 n \ln n + \sum_{j=1}^{\rho} \mu_j n^{(\rho+1-j)/\rho}, \tag{6}$$

$$s(n) \equiv s(\rho; n) := n^{\theta} \sum_{j=0}^{t} (\ln n)^j n^{r_t - j/\rho} q_j(\rho; n), \tag{7}$$

$$q_j(\rho; n) \equiv q_j(n) := \sum_{s=0}^{\infty} b_{sj} n^{-s/\rho} \tag{8}$$

where ρ, r_j, $\mu_0 \rho$ are integers, $\rho \geqslant 1$, μ_j, θ, b_{sj} are complex, $b_{0j} \neq 0$ unless $b_{sj} \equiv 0$ for $s = 0, 1, 2,...$, $r_0 = 0$, $-\pi \leqslant \operatorname{Im} \mu_1 < \pi$.

DEFINITION 1. The series (5), called a *formal series* (FS), will be called a *formal series solution* (FSS) of (1) if, when it is substituted in (1), the equation is divided by $e^{Q(n)}$ and the obvious algebraic manipulations are carried out (see below), then the coefficient of each quantity

$$n^{\theta + r/\rho + s/\omega} (\ln n)^j, \qquad j = 0, 1,..., t, \quad r, s = 0, \pm 1, \pm 2,...,$$

is equal to zero.

A concept of formal equality of two FS can be defined by requiring that, when the series are written with the same value of ρ (as is always possible), then the parameters t, θ, b_{sj}, r_j, μ_j for both series are the same.

The construction of FSS may be carried out by using the identities

$$\begin{aligned} e^{\mu(n+v)^{\delta}} &= e^{\mu n^{\delta}} \left\{ 1 + \sum_{k,j=1}^{\infty} a_{kj} n^{k(\delta-1)+1-j} \right\}, \\ (n+v)^{\mu_0(n+v)} &= n^{\mu_0 n + \mu_0 v} e^{\mu_0 v} \left\{ 1 + \frac{\mu_0 v^2}{2n} + \cdots \right\}, \\ (n+v)^{\alpha} &= n^{\alpha} \left\{ 1 + \frac{\alpha v}{n} + \cdots \right\}, \\ [\ln(n+v)]^r &= \left[\ln n + \frac{v}{n} - \frac{v^2}{2n} + \cdots \right]^r, \end{aligned} \tag{9}$$

although, in practice, it may be difficult to obtain this way other than the first few terms.

DEFINITION 2. Let $f(n)$ be defined for $n = 0, 1, 2,...$. We write

$$f(n) \sim e^{Q(n)} s(n), \qquad n \to \infty, \tag{10}$$

and say that the expansion on the right is the *Birkhoff asymptotic expan-*

sion or *Birkhoff series* for $f(n)$, if for every $k \geqslant 1$ there are bounded functions $A_{kj}(n)$, $j = 0, 1, \ldots, t$, such that

$$e^{-Q(n)} n^{-\theta} f(n) = \sum_{j=0}^{t} (\ln n)^j n^{r_t - j/\rho} \sum_{s=0}^{k-1} b_{sj} n^{-s/\rho} + n^{-k/\rho} \sum_{j=0}^{t} (\ln n)^j n^{r_t - j/\rho} A_{kj}(n).$$

Note that two different functions may possess the same Birkhoff asymptotic expansion, but any given function possesses at most one such expansion. This is because zero has no nontrivial representation of the form (10).

DEFINITION 3. Let

$$W_k := |e^{Q_{j+1}(n+i)} s_{j+1}(n+i)|, \qquad i, j = 0, 1, \ldots, k-1.$$

Then, as is easily verified using the relationships (9), W_k is a FS and

$$W_k = \exp\left\{\sum_{j=1}^{k} Q_j(n)\right\} \bar{s}(n) := e^{\bar{Q}(n)} \bar{s}(n).$$

We say the k FS $\{e^{Q_j(n)} s_j(n)\}$ are *formally linearly independent* if $\bar{s}(n) \neq 0$. Otherwise they are *formally linearly dependent.* We say that *there exist exactly* r FSS of a certain type (e.g., with $Q(n) \equiv 0$) if r FS of that type can be constructed which are formally linearly independent and any $r+1$ such FSS are formally linearly dependent.

Two problems of utmost importance concerning the difference equation (1) are:

(A) Does the equation always possess exactly σ FSS of the general type (10)?

(B) If so, what asymptotic relationship do these FSS bear to the members of some given fundamental set for the equation?

Problem (A) was attacked and partially answered by a number of mathematicians. See Adams [1] and the references given there. However, it was only with the advent of two massive, often overlooked and extraordinarily difficult papers, the first (1930) by George Birkhoff [4] and the second (1932) by Birkhoff and his student W. J. Trjitzinsky [5], that the theory was completed. The results of these two writers yield the

THEOREM 1 (Birkhoff–Trjitzinsky). *There exist exactly σ FSS of Eq.* (1) *of type* (5), *where $\rho = \nu\omega$ for some integer $\nu \geqslant 1$, and each FSS represents asymptotically some solution of the equation* (*in the sense of Definition* 2). *The FSS are, up to multiplicative constants, unique, and the σ solutions so represented constitute a fundamental set for the equation.*

Problem (B) is the problem of using these FSS to construct Birkhoff series (or possibly, linear combinations of Birkhoff series) which represent asymptotically a given solution of the equation. The theory for doing this is very incomplete compared to the theory of problem (A). Though there are a few methods which may serve to reduce the size of the formal subspace in which a desired solution lies, they are not very general and the solution of the problem often has to rely on ad hoc arguments and a specialized knowledge of the properties of the given solution. Lacking such knowledge, the problem is usually unsolvable. In particular, a knowledge of the pointwise values of the given solution does not usually constitute sufficient information. However, in sequences that arise in combinatorics we *do* know, often, that they are increasing and so we are able to rule out a representation by subdominant solutions.

Often shortcuts are possible in the construction of FSS. Since (1) and (3) are, formally, unchanged by the substitution $n \to ne^{2k\pi i\omega}$, other FSS corresponding to a $\rho \to \omega$ can be generated by this substitution. Also, no series containing a term $(\ln n)^k$, $k \geqslant 1$, occurs without associated series containing terms $(\ln n)^j$, $0 \leqslant j \leqslant k-1$. One should first attempt the construction of FSS without the use of logarithmic terms, i.e., assume

$$y(n) := E(n)\,K(n),$$

$$E(n) := e^{\mu_0 n \ln n + \mu_1 n} n^{\theta},$$

$$K(n) := \exp\{\alpha_1 n^{\beta} + \alpha_2 n^{\beta - 1/\rho} + \cdots\}, \qquad \alpha_1 \neq 0, \quad \beta = j/\rho, \quad 0 < j < \rho. \tag{11}$$

Then we find

$$\frac{y(n+k)}{y(n)} = n^{\mu_0 k} \lambda^k \left\{1 + \frac{k\theta + k^2\mu_0/2}{n} + \cdots\right\}$$
$$\times \exp\left\{\alpha_1 \beta k n^{\beta-1} + \alpha_2\left(\beta - \frac{1}{\rho}\right) k n^{\beta - 1/\rho - 1} + \cdots\right\}, \tag{12}$$
$$\lambda := e^{\mu_0 + \mu_1}.$$

We give several examples of the construction.

EXAMPLE 2.1 (Involutions). Let $t(n)$ be the number of involutions on $\{1,..., n\}$ (see Example 1.3). Then $t(n)$ satisfies

$$(n+1)\, t(n)+t(n+1)-t(n+2)=0$$

(the $t(n)$ are related to Hermite polynomials). Making the substitution (12) gives the formal equality

$$\begin{aligned}(n+1)&+n^{\mu_0}\lambda\left\{1+\frac{\theta+\mu_0/2}{n}+\cdots\right\}\\ &\times\exp\left\{\alpha_1\beta n^{\beta-1}+\alpha_2\left(\beta-\frac{1}{\rho}\right)n^{\beta-1/\rho-1}+\cdots\right\}\\ &-n^{2\mu_0}\lambda^2\left\{1+\frac{2\theta+2\mu_0}{n}+\cdots\right\}\\ &\times\exp\left\{2\alpha_1\beta n^{\beta-1}+2\alpha_2\left(\beta-\frac{1}{\rho}\right)n^{\beta-1/\rho-1}+\cdots\right\}=0.\end{aligned}$$

Obviously, only one value of μ_0 will do, namely, $\mu_0=\frac{1}{2}$. But $\mu_0\rho$ must be an integer, and this means $\rho=2$ (if it were greater, ρ (>2) FSS would result on the substitution $n\to ne^{2k\pi i}$, $k=0, 1, 2,..., \rho-1$, giving too many). We find $\beta=\frac{1}{2}$, $\lambda=\pm 1$, $\theta=0$. The two FS are

$$f^{(1)}(n)=\left(\frac{n}{e}\right)^{n/2}e^{\sqrt{n}}\left\{1+\frac{c_1}{\sqrt{n}}+\frac{c_2}{n}+\cdots\right\},$$
$$f^{(2)}(n)=\left(\frac{n}{e}\right)^{n/2}(-1)^n e^{-\sqrt{n}}\left\{1-\frac{c_1}{\sqrt{n}}+\frac{c_2}{n}-\cdots\right\}.$$

Since $t(n)$ is monotone increasing to ∞, we have

$$t(n)\sim Kf^{(1)}(n), \qquad n\to\infty,$$

for some constant $K>0$. It is easy to find $c_1=7/24$ and also to determine $c_2, c_3,\ldots$. Compare [11, p. 67].

EXAMPLE 2.2. Construct FSS for the second-order equation

$$\begin{aligned}(n+b)(n+c)\,y(n)&-(n+1)[(2n+b+c+1)+z]\,y(n+1)\\ &+(n+1)(n+2)\,y(n+2)=0, \qquad z\neq 0. \qquad (13)\end{aligned}$$

Making the substitutions (11), (12) we find the equation to be formally satisfied is

$$1 - 2n^{\mu_0}\lambda\left[1 + \frac{1}{n}\left(\theta + \frac{\mu_0 + 3 + z - b - c}{2}\right) + O(n^{-2})\right]$$

$$\times \exp\left\{\alpha_1 \beta n^{\beta-1} + \alpha_2\left(\beta - \frac{1}{\rho}\right) n^{\beta-1/\rho-1} + \cdots\right\}$$

$$+ n^{2\mu_0}\lambda^2\left[1 + \frac{1}{n}(2\theta + 2\mu_0 + 3 - b - c) + O(n^{-2})\right]$$

$$\times \exp\left\{2\alpha_1 \beta n^{\beta-1} + 2\alpha_2\left(\beta - \frac{1}{\rho}\right) n^{\beta-1/\rho-1} + \cdots\right\} = 0.$$

Obviously, $\mu_0 = 0$ and this implies $\lambda = 1$. This further requires $\rho \neq 1$ or else $z = 0$. Expanding the exponentials in power series gives

$$-z/n + \alpha_1^2 \beta^2 n^{2\beta-2} + (\text{lower-order terms}) = 0.$$

Unless $\beta = \frac{1}{2}$ either $\alpha_1 = 0$ or $z = 0$. Thus $\beta = \frac{1}{2}$, $\alpha_1 = 2\sqrt{z}$. Equating coefficients of $n^{-3/2}$ then yields

$$\theta = \frac{b+c}{2} - \frac{5}{4}.$$

Another series may obtained from this one by letting $n \to ne^{2\pi i}$. The FSS are thus

$$f^{(1)}(n) := n^{(b+c)/2-5/4} e^{-2\sqrt{nz}}\{1 + c_1/\sqrt{n} + c_2/n + \cdots\},$$
$$f^{(2)}(n) := n^{(b+c)/2-5/4} e^{2\sqrt{nz}}\{1 - c_1/\sqrt{n} + c_2/n + \cdots\}.$$

As shown in Wimp [17], the functions

$$y^{(1)}(n) := \frac{(b)_n (c)_n}{n!} \Psi(n+b, b+1-c; z),$$
$$y^{(2)}(n) := \frac{(b)_n}{n!} \Phi(n+b, b+1-c; z) \tag{14}$$

are a fundamental set for the equation, where Ψ, Φ denote confluent hypergeometric functions (see Ref. [7, Chap. 6]) and $(\alpha)_n$ is Pochhammer's symbol

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \qquad n \geqslant 1,$$
$$= 1, \qquad n = 0.$$

Without additional information there is no way to relate these functions to the FSS. However, a result given in Slater [14, p. 80] provides an asymptotic estimate of $y^{(1)}(n)$ and allows us to conclude

$$y^{(1)}(n) \sim Kf^{(1)}(n), \qquad n \to \infty, \quad |\arg z| < \pi$$

for some constant K. (In fact, K can be obtained explicitly from Slater's result.)

EXAMPLE 2.3. Construct FSS for the third-order difference equation

$$xy(n) + (n+2)\, y(n+1) - 2y(n+3) = 0, \qquad x \neq 0.$$

Proceeding as before we find

$$\begin{aligned} x + n^{\mu_0 + 1}\lambda & \left[1 + \frac{1}{n}\left(\theta + 2 + \frac{\mu_0}{2}\right) + \cdots \right] \\ & \times \exp\left\{ \alpha_1 \beta n^{\beta - 1} + \alpha_2 \left(\beta - \frac{1}{\rho}\right) n^{\beta - 1/\rho - 1} + \cdots \right\} \\ & - 2n^{3\mu_0}\lambda^3 \left[1 + \frac{(3\theta + (9/2)\,\mu_0}{n} + \cdots \right] \\ & \times \exp\left\{ 3\alpha_1 \beta n^{\beta - 1} + 3\alpha_2 \left(\beta - \frac{1}{\rho}\right) n^{\beta - 1/\rho - 1} + \cdots \right\} = 0. \end{aligned}$$

A possible value of μ_0 is $\frac{1}{2}$. This gives $\lambda = 2^{-1/2}$ and so we must have

$$-2\theta/n - 2\alpha_1 \beta n^{\beta - 1} - 4\alpha_1^2 \beta^2 n^{2\beta - 2} + (\text{lower-order terms}) = 0.$$

This requires $\beta = 0$, $\theta = 0$. Since $\mu_0 \rho$ must be an integer, we see ρ must be even for this μ_0; obviously, $\rho = 2$, for, if $\rho \geqslant 4$, too many distinct FSS solutions would result on the substitution $n \to ne^{2k\pi i}$, $k = 1, 2, 3, \ldots$. We get the series

$$f^{(1)}(n) := n^{n/2}(2e)^{-n/2}\left(1 + \frac{c_1}{\sqrt{n}} + \frac{c_2}{n} + \cdots\right).$$

Letting $n \to ne^{2\pi i}$ gives another FSS

$$f^{(2)}(n) := n^{n/2}(-1)^n(2e)^{-n/2}\left(1 - \frac{c_1}{\sqrt{n}} + \frac{c_2}{n} - \cdots\right).$$

There is one more series, the one corresponding to $\mu_0 = -1$. For this series we must have $\rho = 1$, or else the substitution $n \to ne^{2\pi i}$ would give yet another FSS, making four in all, i.e., one too many. A quick computation gives the FSS

$$f^{(3)}(n) := n^{-n}(-xe)^n n^{-3/2}\left[1+\frac{d_1}{n}+\frac{d_2}{n^2}+\cdots\right].$$

As is shown by Cole and Pescatore [6], the preceding difference equation is satisfied by the function

$$T(n;x) := \int_0^\infty t^n e^{-t^2-(x/t)}\,dt.$$

In this case, the problem of connecting constants can be solved by applying Laplace's method (see de Bruijn [20, Chap. 4]) to the integral. One finds in fact that

$$T(n;x) \sim \sqrt{\pi/2}\, f^{(1)}(n), \qquad n \to \infty.$$

EXAMPLE 2.4. Consider

$$(n+b)^2 y(n) - (n+1)(2n+2b+1)\, y(n+1) + (n+1)(n+2)\, y(n+2) = 0.$$

Note that this is Eq. (13) with $b=c$ and $z=0$. It is easily seen that a series of the form $q(1;n)\, n^\theta$ is a FSS. (This means that no other FSS with $\rho>1$ is possible.) Substituting this results in a FS whose n^{-1} coefficient vanishes identically and whose n^{-2} coefficient when set equal to zero gives

$$[\theta+(1-b)]^2=0.$$

So there is a single choice of θ, namely, $\theta=b-1$. Logarithmic solutions must occur, since subtracting appropriate multiples of two such equations will result in a solution with a value of θ one less. As a matter of fact, the highest power of $\ln n$ entering in (10) is one. It may be verified directly that a fundamental set for the equation is

$$y^{(1)}(n) := \frac{(b)_n}{n!}, \qquad y^{(2)}(n) := \frac{\psi(n+b)(b)_n}{n!}$$

where ψ is the logarithmic derivative of the gamma function. The FSS may also be obtained from the well-known asymptotic formulas for these functions. In fact, we get

$$y^{(1)}(n) \sim \frac{n^{b-1}}{\Gamma(b)}\left[1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right],$$

$$y^{(2)}(n) \sim \frac{n^{b-1}}{\Gamma(b)}\left[\ln n\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right)+\left(\frac{d_1}{n}+\frac{d_2}{n^2}+\cdots\right)\right].$$

The FS on the right are, of course, the FSS sought. Note the occurrence of the first series in the second. This is no accident. In fact, as the following result of Wimp shows [16], only the logarithmic solution containing the highest power of the logarithm need be determined, and from this one the others may be obtained.

THEOREM 2. *Let*

$$\phi^{(k)}(n) := e^{Q(\rho;n)} n^{\theta} \sum_{l=0}^{k} (\ln n)^l \, q_{t+l-k}(\rho; n) \, n^{r_{k-l}/\rho} \times \frac{(t+1-k)_l}{l!}, \qquad 0 \leqslant k \leqslant t,$$

r_j *an integer.*

Then if $\phi^{(t)}(n)$ *is a* FSS, *so are* $\phi^{(k)}(n)$, $0 \leqslant k \leqslant t-1$.

3. EXAMPLES OF COMBINATORIAL INTEREST

EXAMPLE 3.1. Let $y(n)$ be the number of permutations whose cycles are all of length $\leqslant r$; then (Example 1.4) $y(n)$ satisfies

$$y(n+r) - \sum_{j=1}^{r} (n+r-1)_{j-1} \, y(n+r-j) = 0.$$

We take $r > 2$.

Making the usual substitution gives

$$n^{\mu_0 r} \lambda^r \left[1 + \frac{r\theta + r^2\mu_0/2}{n} + \cdots\right] \exp\left\{\alpha_1 \beta r n^{\beta-1} + \alpha_2 \left(\beta - \frac{1}{\rho}\right) r n^{\beta - 1/\rho - 1} + \cdots\right\}$$

$$- \sum_{j=1}^{r} n^{\mu_0(r-j)+j-1} \lambda^{r-j}$$

$$\times \left[1 + \frac{(j-1)(r+j/2-2) + \theta(r-j) + (r-j)^2 \mu_0/2}{n} + \cdots\right]$$

$$\times \exp\left\{\alpha_1 \beta(r-j) \, n^{\beta-1} + \alpha_2 \left(\beta - \frac{1}{\rho}\right)(r-j) \, n^{\beta - 1/\rho - 1} + \cdots\right\} = 0.$$

Obviously, $\mu_0 > 0$. If $\mu_0 \geqslant 1$, we write

$$n^{\mu_0(r-j)+j-1} = n^{j(1-\mu_0) + r\mu_0 - 1}$$

and observe that the maximum power of n is μ and occurs in a single term, which cannot be. Thus $\mu_0 < 1$. Equating the two highest powers of n gives

$$\mu_0 = \frac{r-1}{r}.$$

Then $\beta = (r-1)/r$ and $\rho = r$. Also $\lambda^2 - 1 = 0$.

Choose $\lambda = 1$, for all the other series may be obtained from this one. Equating powers of $n^{-1/r}$ gives $\alpha_1(r-1) - 1 = 0$ or

$$\alpha_1 = 1/(r-1).$$

Equating powers of $n^{-2/r}$ gives $(\alpha_1^2/2)(r-1)^2 + \alpha_2(r-2) - 1 - \alpha_1\beta = 0$ or

$$\alpha_2 = \frac{r+2}{2r(r-2)},$$

etc. The formal series is

$$y(n) = \left(\frac{n}{e}\right)^{n((r-1)/r)} n^\theta \exp\{\alpha_1 n^{(r-1)/r} + \alpha_2 n^{(r-2)/r} + \cdots\}$$
$$\times \{1 + c_1 n^{-1/r} + c_2 n^{-2/r} + \cdots\},$$

and $r-1$ other formal series may be obtained from this one by the substitution $n \to ne^{2k\pi i}$, $k = 1, 2, \ldots, r-1$. However, all of these are subdominant to $f(n)$ and, since it is clear that $f(n)$ is monotone increasing and none of these other series are, we must have the asymptotic expansion

$$y(n) \sim Kf(n), \qquad n \to \infty,$$

for some constant $K > 0$.

EXAMPLE 3.2: *The Apéry Sequence.* The Apéry sequence (Van der Poorten [21]) satisfies

$$(n+2)^3 u(n+2) - (34n^3 + 153n^2 + 231n + 117)\, u(n+1) + (n+1)^3 u(n) = 0,$$

$$u(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

There is a single value of μ_0, namely, 0, and two values of λ. Thus $\rho = 1$. We find

$$\lambda = 17 \pm 12\sqrt{2}.$$

Equating coefficients of n^{-1} gives $\theta=-\frac{3}{2}$. Since $u(n)$ is monotone increasing, we must have

$$u(n)\sim K(17+12\sqrt{2})^n\, n^{-3/2}(1+c_1/n+\cdots), \qquad n\to\infty,$$

for some constant $K>0$.

This agrees with [21, p. 199].

EXAMPLE 3.3: *Binomial Sums.* The sequence [21, p. 202, footnote 9]

$$u(n)=\sum_{k=0}^{n}\binom{n}{k}^3$$

satisfies

$$(n+2)^2\,u(n+2)-(7n^2+21n+16)\,u(n+1)-8(n+1)^2\,u(n)=0.$$

Proceeding as in the previous example gives

$$u(n)\sim K\frac{8^n}{n}\left(1+\frac{c_1}{n}+\cdots\right), \qquad n\to\infty,$$

for some $K>0$.

The sequence

$$u(n)=\sum_{k=0}^{n}\binom{n}{k}^4$$

satisfies

$$(n+2)^3\,u(n+2)-12(n+\tfrac{3}{2})(n^2+3n+\tfrac{7}{3})\,u(n+1)$$
$$-\,64(n+\tfrac{3}{4})(n+1)(n+\tfrac{5}{4})\,u(n)=0.$$

As before, we find

$$u(n)\sim K\,16^n\,n^{-3/2}(1+c_1/n+\cdots), \qquad n\to\infty,$$

for some $K>0$. Of course for such simple binomial sums it may be easier to use the standard method (Bender [3, pp. 488–489]), but the finer asymptotics, that is, c_1, c_2, etc., are more easily derived by the present method. Of course one drawback of the Birkhoff method is that we do not get K, if we want to express it in terms of known quantities like e and π. But by making an appropriate change of dependent variables, namely, $v(n)=(u(n)/16^n)\,n^{3/2}$, we can get a recurrence for $v(n)$ and since $v(n)\to K$ as $n\to\infty$, we can find K, at least numerically.

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