Discrete Mathematics 49 (1984) 319
North-Holland

NOTE

A COMBINATORIAL PROOF OF NEWTON'S IDENTITIES

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Received 8 April 1983

We are going to give a new proof of Newton's celebrated identities

$$\sum_{r=0}^{k-1}(-1)^r\Big(\sum_{1\leq i_1<\cdots<i_r\leq n} x_{i_1}\cdots x_{i_r}\Big)\Big(\sum_{j=1}^{n} x_j^{k-r}\Big) + (-1)^k\Big(\sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\cdots x_{i_k}\Big)k=0, \quad (*)$$

where n and k are positive integers and $x_1,\ldots,x_n$ are commuting indeterminates.

Consider the set $\mathscr{A}=\mathscr{A}(n,k)$ of pairs (A,j^l) where

(i) A is a subset of $\{1,\ldots,n\}$,
(ii) j is a member of $\{1,\ldots,n\}$,
(iii) $|A|+l=k$, where $|A|$ denotes the number of elements of A,
(iv) $l\geq 0$ and if $l=0$, then $j\in A$.

Define the *weight* of (A,j^l), $w(A,j^l)$ by $w(A,j^l)=(-1)^{|A|}(\prod_{a\in A}x_a)x_j^l$, for example $w(\{1,3,5\},2^3)=(-1)^3x_1x_3x_5\cdot x_2^3=-x_1x_2^3x_3x_5$. It is readily seen that the l.h.s. of $(*)$ is the sum of all the weights of the elements of $\mathscr{A}$. We will now prove that this sum is zero. To this end introduce the mapping $T:\mathscr{A}\to\mathscr{A}$ defined by

$$T(A,j)=\begin{cases}(A/\{j\},j^{l+1}), & j\in A,\\ (A\cup\{j\},j^{l-1}), & j\notin A.\end{cases}$$

This mapping satisfies $w(T(A,j^l))=-w(A,j^l)$ and is an involution (i.e. $T^2=\text{identity}$). Thus all the weights can be arranged in mutually cancelling pairs and their sum is therefore zero.