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NOTE

ANDRE'S REFLECTION PROOF GENERALIZED TO THE MANY-CANDIDATE BALLOT PROBLEM

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··· There is a war between the odd and the even.
(Leonard Cohen)

1. Introduction

Candidates $1,\ldots,n$ received $m_1,\ldots,m_n$ votes respectively, where $m_1 \geqslant \cdots \geqslant m_n \geqslant 0$. The ballot problem asks for the number of ways to count these votes in such a way that at all times the partial scores satisfy $x_1 \geqslant \cdots \geqslant x_n$. This problem was first solved by Frobenious and MacMahon [6, p. 133]. The special case $n=2$ was first solved by Bertrand [3] and received an elegant combinatorial proof by André [1]. An updated exposition of this topic can be found in Knuth [5]. In this paper we extend André's argument to the higher dimensional case. Previous attempts were restricted to $n=3$ [4] and the strict ballot problem [2].

Note that the ballot problem is equivalent to the problem of counting the number of lattice paths, with positive unit steps, from $\mathbf{0}$ to $(m_1,\ldots,m_n)$ such that all the points lie in $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$.

2. Results

Let $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$ be two lattice points, and let $F(a \to b)$ denote the number of lattice walks, with positive unit steps, from a to b. It is well known and easy to see that

$$F(a \to b) = (b_1+\cdots+b_n-a_1-\cdots-a_n)!/(b_1-a_1)!\cdots(b_n-a_n)!. \tag{1}$$

A walk from a to b is *good* if it does not touch any of the hyperplanes $x_i - x_{i+1} = -1$ $(i=1,\ldots,n-1)$, otherwise it is *bad*. Let $G(a \to b)$, $B(a \to b)$ denote the number of good walks and bad walks respectively. For a permutation

$\pi \in S_n$ let $e_\pi = (1-\pi(1), \ldots, i-\pi(i), \ldots, n-\pi(n))$. We claim that

$$\sum_{\pi \text{ even}} B(e_\pi \to m) = \sum_{\pi \text{ odd}} B(e_\pi \to m). \tag{2}$$

Consider a typical bad walk from e_π to m. Let $x_i - x_{i+1} = -1$ be the first hyperplane it meets. Replace the portion of the walk until that meeting by its reflection with respect to $x_i - x_{i+1} = -1$, getting a certain walk from

$$e_{(i,i+1)\pi} = (1-\pi(1), \ldots, i-\pi(i+1),\ i+1-\pi(i), \ldots, n-\pi(n))$$

to m. Since this defines a bijection between

$$\bigcup_{\pi \text{ even}} B(e_\pi \to m) \quad \text{and} \quad \bigcup_{\pi \text{ odd}} B(e_\pi \to m),$$

(2) follows.

Now for $m = (m_1, \ldots, m_n)$ with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$ (i.e. m lies on the positive side of all hyperplanes),

$$F(e_\pi \to m) = B(e_\pi \to m), \quad \text{if } \pi \neq \text{identity}, \tag{3}$$

since e_π lies on the negative side of at least one of the hyperplanes. Thus

$$\begin{aligned} G(0 \to m) &= F(0 \to m) - B(0 \to m) \\ &\overset{(2)}{=} F(0 \to m) + \sum_{\pi \neq \text{id}} (-1)^{\sigma(\pi)} B(e_\pi \to m) \\ &\overset{(3)}{=} \sum_{\pi \in S_n} (-1)^{\sigma(\pi)} F(e_\pi \to m) \\ &\overset{(1)}{=} \sum_{\sigma \in S_n} (-1)^{\sigma(\pi)} \frac{(m_1 + \cdots + m_n)!}{(m_1 - 1 + \pi(1))! \cdots (m_n - n + \pi(n))!} \\ &= (m_1 + \cdots + m_n)! \det(1/(m_i - i + j)!) \\ &= (m_1 + \cdots + m_n)! \prod_{1 \leqslant i < j \leqslant n} (m_i - m_j + j - i)/(m_1 + n - 1)! \cdots m_n! \end{aligned}$$

when the determinant is evaluated. This is the solution of the ballot problem. The determinant is evaluated in the same way as Vandermonde's determinant.

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