Europ. J. Combinatorics (1983) 4, 221–223

A Direct Combinatorial Proof of a Positivity Result

J. Gillis and D. Zeilberger

"A simple result deserves a simple proof"
(Donald E. Knuth, [6] p. 63)

Objects lying in four different boxes are rearranged in such a way that the number of objects in each box stays the same. Askey, Ismail, and Koornwinder proved that the cardinality of the set of rearrangements for which the number of objects changing boxes is even exceeds the cardinality of the set of rearrangements for which that number is odd. We give a simple counting proof of this fact.

Members of four different clubs, each wearing a hat with an insignia of his club, hang their hats on entering the hall. When they leave there is a power failure and the departing guests scramble for hats in the dark. Assuming the hats were picked at an entirely random fashion, would you bet that the number of guests wearing hats with wrong insignias is even?

Askey, Ismail, and Koornwinder [1, p. 285] proved that the answer is always *yes*, no matter how many members belong to each club. Their proof was "analytical" in that it employed an inequality of Koornwinder [7] concerning integrals of products of Laguerre polynomials. Ismail and Tamhankar [5], and independently Gillis and Kleeman [4], gave elementary proofs of this result. However, both of these employed the rather deep "Master Theorem" of McMahon and the relatively sophisticated notion of "generating function". We are going to give a direct counting argument which should be understood by the proverbial bright grade school student.

Although our proof is formally from scratch, it does employ the elegant methods of Foata [2], [3]. As a matter of fact it was conceived while we were reading through Knuth's [6, 5.1.2] lively rendition of Foata's ideas.

Let $\mathcal{M}(a, b, c, d)$ be the set of *multiset permutations* on the multiset $\{1^a 2^b 3^c 4^d\}$ written in the two-line notation (cf [6, p. 24]). These are entities of the form

$$\begin{matrix} \overbrace{1 \cdots 1}^{a} & \overbrace{2 \cdots 2}^{b} & \overbrace{3 \cdots 3}^{c} & \overbrace{4 \cdots 4}^{d} \\ i_1 \cdots i_a & j_1 \cdots j_b & k_1 \cdots k_c & l_1 \cdots l_d \end{matrix}$$

where the bottom is a rearrangement of the top. The *weight* $w(\pi)$ of the multiset permutation π is defined to be $(-1)^{U_\pi}$ where U_π is the number of columns of the form $\begin{matrix} x \\ y \end{matrix}$, $x \neq y$. For example, if $\pi = \begin{matrix} 111122333444 \\ 441223133411 \end{matrix}$ then $U_\pi = 7$ and $w(\pi) = (-1)^7 = -1$. If $\mathcal{U}$ is a set, we write $w(\mathcal{U}) = \sum_{u \in \mathcal{U}} w(u)$, and if $\mathcal{U}$ and $\mathcal{V}$ are sets, we denote as usual $\mathcal{U} \times \mathcal{V} = \{(u, v);\ u \in \mathcal{U},\ v \in \mathcal{V}\}$.

Theorem ([4]).

$$w(\mathcal{M}(a, b, c, d)) = \sum_u \frac{(a+b-u)!\,(c+d-u)!\,u!^2}{a!\,b!\,c!\,d!} \left\{ \sum_r (-1)^r \binom{a}{r} \binom{b}{u-r} \sum_s (-1)^s \binom{c}{s} \times \binom{d}{u-s} \right\}^2 \tag{1}$$

and thus is positive.

0195–6698/83/030221+03 $02.00/0

We first prove:

LEMMA 1. *Let $\mathscr{F}(a, b, c)$ be the set of multiset permutations on $\{1^a 2^b 3^c\}$ having no occurrences of* $\genfrac{}{}{0pt}{}{3}{3}$. *Then $\mathscr{M}(a, b, c, d) \leftrightarrow \bigcup_u \mathscr{F}(a, b, u) \times F(c, d, u)$ and if $\pi \leftrightarrow (\sigma_1, \sigma_2)$ then $w(\pi) = w(\sigma_1) w(\sigma_2)$.*

PROOF. Given π, we obtain σ_1 by replacing all 4's in π by 3's and discarding all columns $\genfrac{}{}{0pt}{}{3}{3}$; similarly we define σ_2 by replacing all 1's by 2's and discarding all columns $\genfrac{}{}{0pt}{}{2}{2}$. For example

$$\begin{matrix}1111222333444\\3142144231321\end{matrix} \leftrightarrow \left(\begin{matrix}11112223333\\31321332121\end{matrix}, \begin{matrix}2222333444\\3444232322\end{matrix}\right).$$

Clearly $\sigma_1 \in \mathscr{F}(a, b, u)$ in $\{1^a, 2^b, 3^u\}$ and $\sigma_2 \in \mathscr{F}(c, d, u)$ in $\{3^c 4^d 2^u\}$ for some u. It is readily seen that (σ_1, σ_2) contains just enough information to get π back and we thus have a bijection. Also the parity of U_π equals that of $U_{\sigma_1} + U_{\sigma_2}$ and so $w(\pi) = w(\sigma_1) w(\sigma_2)$.

LEMMA 2.

$$w(\mathscr{F}(a, b, c)) = \frac{(a+b-c)!\,c!}{a!\,b!} \left[\sum_{\alpha} (-1)^\alpha \binom{a}{\alpha} \binom{b}{c-\alpha}\right]^2.$$

PROOF. Consider a typical element of $\mathscr{F}(a, b, c)$. If it has α occurrences of $\genfrac{}{}{0pt}{}{1}{3}$ and β occurrences of $\genfrac{}{}{0pt}{}{3}{1}$ then (in the notation of [6, p. 28])

$$\begin{array}{ccc}
\overbrace{1\cdots 1}^{a} & \overbrace{2\cdots 2}^{b} & \overbrace{3\cdots 3}^{c}\\
\multicolumn{2}{c}{\underbrace{\underbrace{\sqcup\quad\sqcup}_{\alpha\ 3\text{'s}}\quad \underbrace{\sqcup\quad\sqcup}_{(c-\alpha)\ 3\text{'s}}}_{\substack{(a-\beta)\ 1\text{'s}\\ (b-c+\beta)\ 2\text{'s}}}} & \underbrace{\sqcup\quad\sqcup}_{\substack{\beta\ 1\text{'s}\\ (c-b)\ 2\text{'s}}}
\end{array}$$

The number of elements of $\mathscr{F}(a, b, c)$ with such specifications is $\binom{a}{\alpha}\binom{b}{c-\alpha}\binom{c}{\beta}\binom{a+b-c}{a-\beta}$. All these have the same weight $(-1)^{\alpha+\beta}$ since the "3-free" part looks as follows (for some γ):

$$\begin{array}{cc}
\overbrace{1\cdots 1}^{a-\alpha} & \overbrace{2\cdots 2}^{b-c+\alpha}\\
\underbrace{\sqcup\quad\sqcup}_{\substack{\gamma\ 1\text{'s}\\ (a-\alpha-\gamma)\ 2\text{'s}}} & \underbrace{\sqcup\quad\sqcup}_{(a-\beta-\gamma)\ 1\text{'s}}
\end{array},$$

and thus

$$U_\pi = \#\begin{smallmatrix}1\\3\end{smallmatrix} + \#\begin{smallmatrix}2\\3\end{smallmatrix} + \#\begin{smallmatrix}3\\1\end{smallmatrix} + \#\begin{smallmatrix}3\\2\end{smallmatrix} + \#\begin{smallmatrix}1\\2\end{smallmatrix} + \#\begin{smallmatrix}2\\1\end{smallmatrix}$$

$$= c + c + a - \alpha - \gamma + a - \beta - \gamma = 2(a + c - \gamma) - \alpha - \beta,$$

and thus $w(\pi) = (-1)^{\alpha+\beta}$.

Summing over all α, β we get

$$w(\mathscr{F}(a, b, c)) = \sum_{\alpha,\beta} (-1)^{\alpha+\beta}\binom{a}{\alpha}\binom{b}{c-\alpha}\binom{a+b-c}{a-\beta}$$

$$= \frac{(a+b-c)!c!}{a!b!}\sum (-1)^{\alpha+\beta}\binom{a}{\alpha}\binom{b}{c-\alpha}\binom{a}{\beta}\binom{b}{c-\beta}$$

$$= \frac{(a+b-c)!c!}{a!b!}\left[\sum (-1)^{\alpha}\binom{a}{\alpha}\binom{b}{c-\alpha}\right]^2.$$

PROOF OF THE THEOREM. From Lemma 1 we have

$$w(\mathscr{M}(a, b, c, d)) = \sum_u w(\mathscr{F}(a, b, u))w(\mathscr{F}(c, d, u))$$

and the theorem follows from Lemma 2.

If $a = b = c = d = N$, say, then it is readily seen that

$$w(\mathscr{M}(N, N, N, N)) = \sum \left[\binom{2N-2V}{N-V}\binom{2V}{V}\right]^2.$$

A routine application of Stirling's formula yields that this is asymptotically $2^{4N+1}\log N/(\pi^2 N)$ (notice that for $N = 2$ this gives 17 compared with the exact value of 22, not bad!). The total number of arrangements is of course $(4N)!/N!^4 \sim 2^{8N-(1/2)}/(\pi N)^{3/2}$. If the probability of an even number of wrong hats is $\frac{1}{2}(1+\alpha)$ then $\alpha \sim (2^{4N+1}\log N/(\pi^2 N))/(2^{8N-(1/2)}/(\pi N)^{3/2}) = 2^{-4N+(3/2)}N^{1/2}\log N/\pi^{1/2}$, hardly big enough to be worth betting on!

REFERENCES

1. R. Askey, M. E. H. Ismail, and T. Koornwinder, Weighted permutation problems and Laguerre polynomials, *J. Combin. Theory, Ser. A* **25** (1978), 277–287.
2. P. Cartier and D. Foata, Problémes combinatoires de commutation et réarrangement, *Lecture Notes in Mathematics No. 89*, Springer, Berlin, 1969.
3. D. Foata, Etude algébrique de certains problèmes d'analyse combinatoire et du calcul de probabilités, *Publ. I. Statist. U. Paris* **14** (1965), 81–241.
4. J. Gillis and J. Kleeman, A combinatorial proof of a positivity result, *Math. Proc. Camb. Phil. Soc.* **86** (1979), 13–19.
5. M. E. H. Ismail and M. V. Tamhankar, A combinatorial approach to some positivity problems, *SIAM J. Math. Anal.* **10** (1979), 478–485.
6. D. E. Knuth, *The Art of Computer Programming, Vol. 3: Sorting and Searching*, Addison-Wesley, Reading, MA, 1973.
7. T. Koornwinder, Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition theorem, *J. London Math. Soc.* **18**, (2) (1978), 101–114.

Received 15 July 1982

J. GILLIS AND D. ZEILBERGER†
Faculty of Mathematics, Weizmann Institute, of Science, Rehovot, Israel

† *Current address: Department of Mathematical Science, Drexel University, Philadelphia, PA 19104, U.S.A.*