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T. L. Hill's Graphical Method for Solving Linear Equations

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T. L. Hill's method for solving homogeneous linear equations by tree enumeration is presented in standard mathematical language and extended to nonhomogeneous systems. It is also shown how Hill's method implies the matrix-tree theorem and the Wang algebra of electrical networks.

1. INTRODUCTION

That tree enumeration can be used for solving systems of linear equations was independently realized by physicists [9], electrical engineers [11], chemists [7] and most recently by biophysicists [4, 5]. Going in the opposite direction, mathematicians realized that one can use linear algebra to count trees ([1, p. 378; 8, p. 578; 10, pp. 39–51]). However, both mathematicians and scientists had to resort to 'fancy' mathematics in order to establish this connection. It is therefore remarkable that the biophysicist T. L. Hill came up with a very elegant and short combinatorial proof [4, 5].

We are going to present Hill's proof in standard mathematical language and show how it implies the matrix-tree theorem. We also extend Hill's method to inhomogeneous equations and show how it justifies the Wang algebra of networks, which was extensively studied by Duffin [3] and Bott and Duffin [2].

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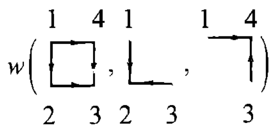
2. Hill's Method and Proof

We refer the reader to [10] for the definitions of the standard terms "directed graph," "rooted tree," "cycle," etc. However, contrary to common usage, we do not allow loops (edges which connect a node to itself) and all the paths in a rooted tree are directed *toward* the root. It is well known [10, p. 3] that a tree with n nodes has $n-1$ edges and that from every node there is a unique path to the root. Thus every node, except the root, has exactly one outgoing edge.

Let α_{ij}, $1 \leqslant i \neq j \leqslant n$, be *commuting* indeterminates. The weight of an edge (i, j) is α_{ij}. The weight, $w(G)$, of a directed graph is defined to be the product of its edge weights. Finally, the weight of a set of directed graphs is the sum of the weights of its members. For example,

$$w(2 \longrightarrow 3) = \alpha_{23}, \qquad w\left(1 \longrightarrow 2 \longleftarrow 3\right) = \alpha_{12}\alpha_{32},$$



$$= \alpha_{12}\alpha_{23}\alpha_{43}\alpha_{14} + \alpha_{12}\alpha_{32} + \alpha_{14}\alpha_{34}.$$

Theorem 1 (Hill [4, 5], King and Altman [7]). *The solution of the homogeneous system*

$$\left(\sum_{j \neq i} \alpha_{ij}\right) x_i = \sum_{j \neq i} \alpha_{ji} x_j, \qquad i = 1, \ldots, n, \tag{1}$$

is given by

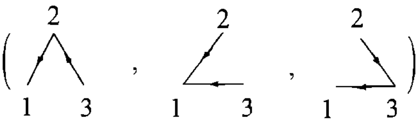
$$x_i = kw(\Gamma_i), \qquad i = 1, \ldots, n \; (k \text{ scalar}), \tag{2}$$

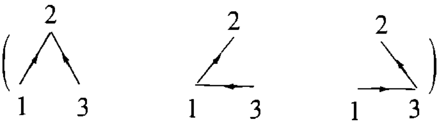
where Γ_i is the set of trees on $\{1, \ldots, n\}$ rooted at i.

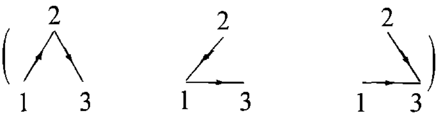
Note. Hill [5] chooses k such that $\Sigma_{i=1}^{n} x_i = 1$.

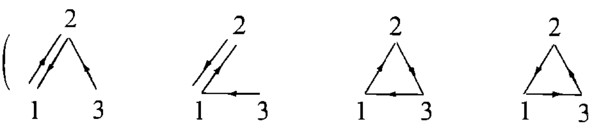
Proof (Hill [5, pp. 201–205]). Let f_i $(i = 1, \ldots, n)$ be the set of connected directed graphs S on $\{1, \ldots, n\}$ such that S has exactly one cycle, i is contained in that cycle, and there is a unique path to the cycle from every node. Now if $T \in \Gamma_i$ adding an edge (i, j) will create an element $S \in f_i$ with $w(S) = \alpha_{ij} w(T)$. Conversely, if $S \in f_i$ and the outgoing edge from i is to j then deleting (i, j) from S will yield an element T of Γ_i with $w(S) = \alpha_{ij} w(T)$. Thus $(\Sigma_{j \neq i} \alpha_{ij})\, w(\Gamma_i) = w(f_i)$ $(i = 1, \ldots, n)$. Similarly, by considering the incoming edge at i, $\Sigma_{j \neq i}\, \alpha_{ji} w(\Gamma_j) = w(f_i)$, establishing that $x_i = w(\Gamma_i)$, $(i = 1, \ldots, n)$ is indeed a solution of (1).

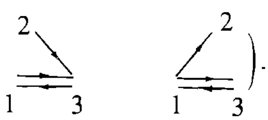
EXAMPLE. For $n = 3$.

$x_1 = w\Big($  $\Big) = \alpha_{32}\alpha_{21} + \alpha_{21}\alpha_{31} + \alpha_{23}\alpha_{31},$

$x_2 = w\Big($  $\Big) = \alpha_{12}\alpha_{32} + \alpha_{31}\alpha_{12} + \alpha_{13}\alpha_{32},$

$x_3 = w\Big($  $\Big) = \alpha_{12}\alpha_{23} + \alpha_{21}\alpha_{13} + \alpha_{13}\alpha_{23},$

$f_1 = \Big($ 

 $\Big).$

It should be noted that Hill's method is particularly useful when one knows beforehand that many α_{ij}'s are zero. Let G be the directed graph on $\{1,\dots, n\}$ which has an edge (i, j) iff $\alpha_{ij} \neq 0$. Then the only surviving terms in $x_i = w(\Gamma_i) = \sum_{T\in\Gamma_i} w(T)$ are those T which are subgraphs of G. Thus $x_i = w(\Gamma_i \cap \mathcal{G})$, where $\mathcal{G}$ is the set of spanning subgraphs of G. Now if one interprets α_{ij} as the number of edges from i to j (allowing multiple edges) then $w(\Gamma_i \cap \mathcal{G})$ counts the number of spanning trees of G which are rooted at i.

Let us write the system (1) in matrix notation

$$(x_1,\dots, x_n)A = 0, \tag{3}$$

where $A = (a_{ij})$ with $a_{ii} = \sum_{j\neq i} \alpha_{ij}$, $a_{ij} = -\alpha_{ij}$. Let A_i be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and ith column. By standard linear algebra, $x_i = \det A_i$ $(i = 1,\dots, n)$ is also a solution of (3). By uniqueness $w(\Gamma_i \cap \mathcal{G}) = k \det A_i$ for some universal constant k, which is easily seen to be 1. Thus $w(\Gamma_i \cap \mathcal{G}) = \det A_i$. We have just given a short combinatorial proof of Borchardt's matrix-tree theorem [1; 8, p. 378], which states that the number of spanning trees rooted at i is $\det A_i$.

3. Inhomogeneous Systems

We will now apply Hill's method to the problem of solving inhomogeneous systems.

THEOREM 2. *The solution of the following system of $n-1$ equations and $n-1$ unknowns*

$$\left(\sum_{i=2}^{n} \alpha_{1i}\right) x_1 - \sum_{j=2}^{n-1} \alpha_{j1} x_j = 1, \tag{4a}$$

$$\left(\sum_{j\neq 1} \alpha_{ij}\right) x_i - \sum_{\substack{j\neq i\\ 1\leq j\leq n-1}} \alpha_{ji} x_j = 0, \qquad i = 2,\ldots, n-1, \tag{4b}$$

is given by

$$x_i = w(\Gamma_{i,1;n})/w(\Gamma_n), \qquad i = 1,\ldots, n-1, \tag{5}$$

where

(i) Γ_n *is the set of trees on* $\{1,\ldots, n\}$ *rooted at* n;

(ii) $\Gamma_{i,1;n}$ *is the set of directed graphs on* $\{1,\ldots, n\}$ *with two components; one component is a tree rooted at i which contains* 1 *and the other component is a tree rooted at n* ($i = 1,\ldots, n-1$).

Proof. Let $f_{i,1;n}$ be the set of two-component directed graphs such that one component is a tree rooted at n and the other component is a directed graph containing 1 and i, having exactly one cycle, i belongs to that cycle and there is a unique path to the cycle from every node of that component. We claim that

$$\left(\sum_{i=2}^{n} \alpha_{1i}\right) w(\Gamma_{1,1;n}) = w(f_{1,1;n}) + w(\Gamma_n). \tag{6}$$

Indeed, adjoining the edge $(1, i)$ to an element $T \in \Gamma_{1,1;n}$ will either create an element of $f_{1,1;n}$ (if i belongs to the component of 1 in T) or an element of Γ_n (if i belongs to the component of n). Similarly,

$$\sum_{i=2}^{n-1} \alpha_{i1} w(\Gamma_{i,1;n}) = w(f_{1,1;n}). \tag{7}$$

Combining (6) and (7) establishes that the solution (5) satisfies (4a).

Now, we also claim that if $\mathcal{U}_i = \{T \in \Gamma_n$; the path from 1 to n passes through $i\}$, then

$$\left(\sum_{\substack{j \neq i \\ 1 \leqslant j \leqslant n-1}} \alpha_{ij}\right) w(\Gamma_{i,1;n}) = w(f_{i,1;n}) + w(\mathcal{U}_i), \qquad i = 2, \ldots, n+1. \tag{8}$$

Indeed, adjoining the edge (i, j) to $T \in \Gamma_{i,1;n}$ either creates an element of $f_{i,1;n}$ (if j belongs to the component of i in T) or an element of $\mathcal{U}_i$ (if j belongs to the component of n). Similarly we have

$$\sum_{\substack{j \neq i \\ 1 \leqslant j \leqslant n-1}} \alpha_{ji} w(\Gamma_{j,1;n}) = w(f_{i,1;n}) + w(\mathcal{U}_i), \qquad i = 2, \ldots, n-1. \tag{9}$$

Combining (8) and (9) establishes that the proposed solution (5) indeed satisfies (4b), completing the proof of the theorem.

Remark 1. As in Section 2, if some α_{ij}'s are zero and G is the graph which has a vertex (i, j) iff $\alpha_{ij} \neq 0$ and $\mathcal{G}$ is the set of spanning subgraphs of G, then $x_i = w(\Gamma_{i,1;n} \cap \mathcal{G})/w(\Gamma_n \cap \mathcal{G})$.

Remark 2. If $A = (a_{ij})$ is the generic $(n-1) \times (n-1)$ matrix, then the system $(x_1, \ldots, x_{n-1})A = (1, 0, \ldots, 0)$ can be written in the form (4) by setting $\alpha_{ij} = -a_{ij}$ $(1 \leqslant i \neq j \leqslant n)$ and $\alpha_{ii} = \sum_{j=1}^{n} a_{ij}$.

Remark 3. Consider the system of linear differential equations

$$\frac{d\mathfrak{N}_i}{dt} = \left(\sum_{j \neq i} \alpha_{ij}\right)\mathfrak{N}_i - \sum_{j \neq i} \alpha_{ji}\mathfrak{N}_j \qquad (i = 1, \ldots, n).$$

In the steady state one sets $d\mathfrak{N}_i/dt = 0$ getting the system (1), the case considered in [5]. One may use Theorem 2 to get diagrammatic solutions of the Laplace transforms, $\hat{\mathfrak{N}}_i(s)$. Indeed, taking Laplace transform and assuming that $\mathfrak{N}_1(0) = 1$, $\mathfrak{N}_2(0) = \cdots = \mathfrak{N}_n(0) = 0$, we get

$$\left(\sum_{j \neq i} \alpha_{ij} - s\right)\hat{\mathfrak{N}}_i - \sum \alpha_{ji}\hat{\mathfrak{N}}_j = \delta_{1i}$$

and Theorem 2 applies with $\{1, \ldots, n+1\}$ and all the edges $(i, n+1)$ having weight $-s$.

Remark 4. We have established that the Hill diagram method constitutes a common algorithm both for kinetic diagram and for electrical network representation of bioenergetic systems as represented in [6].

4. The Wang Algebra

Consider a network with n nodes $1, \ldots, n$ such that the wire between i and j has conductance α_{ij} $(1 \leqslant i \neq j \leqslant n)$. Suppose that there is a battery which causes a current of 1 amp to flow from n to 1 and that n is grounded. If x_i denotes the voltage at i $(i = 1, \ldots, n-1)$ then it is readily seen that applying Kirchoff's current law at every node yields the system (4). Thus the system of linear equations considered in network theory is a special case of (4) where one has in addition that $\alpha_{ij} = \alpha_{ji}$. The value of x_1 has a special significance in network theory, being the joint conductance between node 1 and node n. The electrical engineer K. T. Wang [12] developed an ingenious method for deriving an explicit formula for x_1. This method was justified and extensively studied by Duffin [3] and Bott and Duffin [2]. The following theorem extends Wang's method to the nonsymmetrical case $\alpha_{ij} \neq \alpha_{ji}$.

In $\mathbb{C}[\{\alpha_{ij}\}]$ define the involution $x \to \bar{x}$ by $\overline{\alpha_{ij}} = \alpha_{ji}$ and extend it to be a ring homomorphism: $\overline{xy} = \bar{x}\bar{y}, \overline{x+y} = \bar{x} + \bar{y}$.

Theorem 3. *Suppose that in the system* (4), $\alpha_{ij} \neq 0$ *if and only if* $\alpha_{ji} \neq 0$. *Let G be the graph whose nodes are $\{1, \ldots, n\}$ and whose edges are the (i, j) for which $\alpha_{ij} \neq 0$. An explicit formula for x_1 is obtained by expanding $\prod_{i=2}^{n-1} (\sum_{(i,j) \in G} \alpha_{ij}) / \prod_{i=1}^{n-1} (\sum_{(i,j) \in G} \alpha_{ij})$ and applying the following simplification rules to both numerator and denominator*:

$$\text{(i) } x\bar{x} = 0, \qquad \text{(ii) } x + \bar{x} = 0,$$

in addition to the regular distributive and commutative laws.

Proof. By the remark following the proof of Theorem 2, $x_1 = w(\Gamma_{1,1;n} \cap \mathcal{G}) / w(\Gamma_n \cap \mathcal{G})$. Let $\mathcal{V}_n (\mathcal{V}_{1,n})$ be the set of directed graphs on $\{1, \ldots, n\}$ with the property that n (1 and n) has (have) no outgoing edges and every other node has exactly one outgoing edge. By the remarks made in the beginning of Section 2, $\Gamma_n \subset \mathcal{V}_n$ and $\Gamma_{1,1;n} \subset \mathcal{V}_{1,n}$. We have

$$w(\mathcal{V}_n \cap \mathcal{G}) = \prod_{i=1}^{n-1} \left(\sum_{(i,j) \in G} \alpha_{ij} \right) \quad \text{and} \quad w(\mathcal{V}_{1,n} \cap \mathcal{G}) = \prod_{i=2}^{n-1} \left(\sum_{(i,j) \in G} \alpha_{ij} \right).$$

In order to get $w(\Gamma_n \cap \mathcal{G}) = \sum_{T \in \Gamma_n \cap \mathcal{G}} w(T)$ from $w(\mathcal{V}_n \cap \mathcal{G}) = \sum_{S \in \mathcal{V}_n \cap G} w(S)$ we have to get rid of all the terms $w(S)$ for which S is not a tree, i.e. has a cycle. If S has a cycle of length 2: $\{(i, j), (j, i)\}$ then $w(S)$ is eliminated by simplification rule (i). If it has a cycle of length $\geqslant 3$: $\{(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)\}$ then the directed graph S' obtained from S by reversing that cycle must also be eliminated and rule (ii) gets rid of both $w(S)$ and $w(S')$ at the same time. Similarly, the application of (i) and (ii) to $w(\mathcal{V}_{1,n} \cap \mathcal{G})$ leaves us with $w(\Gamma_{1,n} \cap G)$. ∎

Remark. In the symmetrical case $\alpha_{ij} = \alpha_{ji}$ so $x = \bar{x}$ and rules (i) and (ii) become $x^2 = 0$, $x + x = 0$. These are the original rules of the Wang algebra [3, 4]. Thus the proofs of Theorems 2 and 3 taken together give a simple combinatorial proof of the Wang algebra.

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