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NOTE

ENUMERATION OF WORDS BY THEIR NUMBER OF MISTAKES

Doron ZEILBERGER*

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

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Consider all words in $\{1, \ldots, n\}$. A fixed set of words is labeled as the set of "mistakes". A generating function for the number of words with $m_1$1's, ..., m_n n's and k mistakes is given. This generalizes a result of Gessel who considered the case where all the mistakes are two-lettered. A similar result has been independently obtained by Goulden and Jackson.

1.

Fix an alphabet $\{1, \ldots, n\}$. To every word $w = \sigma_1 \cdots \sigma_l$ we associate the monomial $x^w = x_{\sigma_1} \cdots x_{\sigma_l}$ in the non-commuting indeterminates $x_1, \ldots, x_n$. A *subword of* $\sigma_1 \cdots \sigma_l$ is anything of the form $\sigma_i \sigma_{i+1} \cdots \sigma_j$, $1 \leq i \leq j \leq l$. Let L be a set of words to be labeled as "mistakes". We assume that no proper subword of a mistake is a mistake. The number of subwords of w which belong to L is the *number of mistakes of* w and will be denoted by $d(w)$. For example if $L = \{123, 231\}$, $d(1231) = 2$, because both 123 and 231 belong to L. A word w is said to be of type $(m_1, \ldots, m_n)$ if it has m_1 1's, m_2 2's, ..., m_n n's; e.g. the type of 12112331 is $(4, 2, 2)$. Let M be the set of words w such that every letter of w belongs to some mistake and every mistake, except the last, overlaps, on the right, with another mistake. For example if $L = \{123, 231, 312\}$, $M = \{123, 231, 312, 1231, 2312, 3123, 12312, 23123, 31231, \ldots, \text{etc.}\}$.

The following is a generalization of Theorem 7.2 in Gessel [2]; Gessel's theorem considers the case where L only contains two-lettered words.

Theorem

$$\sum_{w \in \text{all words}} t^{d(w)} x^w = \left[1 - x_1 - \cdots - x_n - \sum_{v \in M} (t-1)^{d(v)} x^v\right]^{-1}. \tag{1}$$

Proof. Let $s(w)$ denote the type of a word w. Let $C(\boldsymbol{m}) = C(m_1, \ldots, m_n)$ be the set of words of type $\boldsymbol{m} = (m_1, \ldots, m_n)$. Define

$$F(\boldsymbol{m}) = \sum_{w \in (\boldsymbol{m})} t^{d(w)} x^w. \tag{2}$$

*Current address: Dept. of Theoret. Math., Weizmann Insitute of Science, Rehovot, Israel.

We shall prove that for $\boldsymbol{m} \neq \boldsymbol{0}$

$$F(\boldsymbol{m}) = \sum_{i=1}^{n} F(\boldsymbol{m}-\boldsymbol{e}_i)x_i + \sum_{v\in M} (t-1)^{d(v)} F(\boldsymbol{m}-s(v))x^v, \tag{3}$$

where $\boldsymbol{e}_i = (0, \ldots, 1, 0, \ldots, 0)$ with the 1 on the ith place.

This will be accomplished by showing that for any $w \in C(\boldsymbol{m})$, the coefficient of x^w in the r.h.s. of (3) is $t^{d(w)}$. Indeed, let w_2 be the maximal tail of w which belongs to M; then $w = w_1 w_2$ for some word w_1, and $d(w) = d(w_1) + d(w_2)$. Note that w has $d(w_2)$ tails which belong to M and thus x^w appear $d(w_2)+1$ times in the r.h.s of (3). Since w loses a mistake by chopping off its last letter and loses $k+1$ mistakes by chopping off a tail which belongs to M and which has k mistakes, the coefficient of x^w in the r.h.s. of (3) is (Put $d(w_1) = d_1$, $d(w_2) = d_2$):

$$t^{d_1}[t^{d_2-1} + (t-1)t^{d_2-2} + (t-1)^2 t^{d_2-3} + \cdots + (t-1)^{d_2-1} + (t-1)^{d_2}] = t^{d_1}t^{d_2} = t^{d(w)}.$$

Here we used (2) with $\boldsymbol{m}$ replaced by $\boldsymbol{m}-\boldsymbol{e}_i$ and $\boldsymbol{m}-s(v)$.

Let $\delta(\boldsymbol{m})$ be the discrete delta function: $\delta(\boldsymbol{0}) = 1$; $\delta(\boldsymbol{m}) = 0$, $\boldsymbol{m} \neq \boldsymbol{0}$. Then, since $F(\boldsymbol{0}) = 1$ and by convention F is zero outside $\mathbf{N}^n$:

$$F(\boldsymbol{m}) - \sum_{i=1}^{n} F(\boldsymbol{m}-\boldsymbol{e}_i)x_i - \sum_{v\in M} (t-1)^{d(v)} F(\boldsymbol{m}-s(v))x^v = \delta(\boldsymbol{m}).$$

Summing both sides over all $\boldsymbol{m} \in \mathbf{Z}^n$ yields

$$\left[\sum_{w\in\text{all words}} t^{d(w)}x^w\right]\left[1 - x_1 - x_2 - \cdots - x_n - \sum_{v\in M} (t-1)^{d(v)}x^v\right] = 1,$$

from which (1) follows.

2. The commutative case

If we let $x_1, \ldots, x_n$ commute in (1) we obtain a generating function for $G(\boldsymbol{m}; k)$, the number of words of type $\boldsymbol{m}$ with exactly k mistakes:

$$\sum G(m_1, \ldots, m_n; k)x_1^{m_1} \cdots x_n^{m_n}t^k = \left[1 - x_1 - \cdots - x_n - \sum_{v\in M} (t-1)^{d(v)}x^v\right]^{-1}. \tag{4}$$

Example. $n = 3$, $L = \{123, 132\}$. Here $L = M$ and

$$\sum G(m_1, m_2, m_3, k)x_1^{m_1}x_2^{m_2}x_3^{m_3}t^k = [1 - x_1 - x_2 - x_3 + 2(1-t)x_1x_2x_3]^{-1}.$$

Putting $t = -1$ we get

coefficient of $x_1^{m_1}x_2^{m_2}x_3^{m_3}$ in $[1 - x_1 - x_2 - x_3 + 4x_1x_2x_3]^{-1} =$
#{words in $C(m)$ with an even number of mistakes}
$-$#{words in $C(m)$ with an odd number of mistakes}.

Askey and Gasper [1] proved that the l.h.s. is positive. It will be nice to give a direct proof that the r.h.s. is positive.

Finally let us mention that whenever L is finite but M is infinite it is still possible to evaluate the sum on the r.h.s. of (4) using the geometric series expansion of a certain matrix: $\sum A^k = (I-A)^{-1}$. Thus whenever L is finite the generating function $\sum G(\boldsymbol{m}; k)\, \boldsymbol{x}^{\boldsymbol{m}} t^k$ is a rational function. The details are left to the sufficiently interested reader.

Remark. The results of this paper have been obtained independently by Goulden and Jackson [3]. We refer the reader to this very interesting paper for detailed applications and algorithms.

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References

[1] R. Askey and G. Gasper, Certain rational functions whose power series have positive coefficients, Amer. Math. Monthly 79 (1972) 337–341.

[2] I. Gessel, Generating Functions and Enumeration of Sequences, Ph.D. thesis, MIT, Cambridge, MA (1977).

[3] I.P. Goulden and D.M. Jackson, An inversion theorem for cluster decompositions of sequences with distinguished subsequences, J. London Math. Soc. (to appear).