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Mathematics

All binomial identities are verifiable

(binomial coefficient identities/hypergeometric series/recurrence relations)

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ABSTRACT Sister Celine Fasenmyer's technique for obtaining pure recurrence relations for hypergeometric polynomials is formalized and used to show that every identity involving sums of products of binomial coefficients can be verified by checking a finite number of its special cases.

The proofs of many results in the theory of hypergeometric series involve intricate series manipulations in which a few basic identities are repeatedly used to obtain more complex identities. Even binomial sums, which are nothing but terminating hypergeometric series, required a variety of *ad hoc* techniques (1). Here I show that the verification of any given binomial identity only requires a finite number of routine checks. The major step in the proof is realizing the far-reaching implication of Sister Celine's technique.

Sister Celine's technique (2, 3) is a systematic way of obtaining pure recurrence relations for families of terminating hypergeometric polynomials. However, its full significance only could have been realized after the appearance of Stanley's paper (4), which considers the class of *P*-recursive sequences, that is, sequences which are solutions of some linear recurrence equation with polynomial coefficients. Full proofs and many generalizations of our results will appear elsewhere.

Definition 1: Let N be the set of nonnegative integers. $F: N^n \to \mathbf{C}$ is multihypergeometric if for $1 \le k \le n$, $F(a_1, \ldots, a_k + 1, \ldots, a_n)/F(a_1, \ldots, a_n)$ is a rational function in $a_1, \ldots, a_n$.

Definition 2: $F: N \to \mathbf{C}$ is *P*-recursive if it satisfies a linear recurrence equation with polynomial coefficients—i.e., if there exist polynomials $P_0(a), \ldots, P_k(a)$, such that for $a \ge k$: $P_0(a)F(a) + P_1(a)F(a-1) + \ldots + P_k(a)F(a-k) = 0$.

THEOREM 1. *Let* F(a,k) *be multihypergeometric; for each* a, *let zero be multihypergeometric except for a finite number of* ks. *Then* $G(a) = \Sigma_{k=-\infty}^{\infty} F(a,k)$ *is* P-*recursive (implicit in Sister Celine's technique)*.

Sketch of Proof: Consider the polynomial $g(a,x) = \Sigma_{k=-\infty}^{\infty} F(a,k)x^k$. Sister Celine's technique (3), which can be formalized and shown to hold in general, implies that $g(a,x)$ satisfies a recurrence of the form $P_0(a,x)g(a,x) + P_1(a,x)g(a-1,x) + \ldots + P_k(a,x)g(a-k,x) = 0$, in which the P_is are polynomials in a and x. Because $G(a) = g(a,1)$, the theorem follows.

$$\textit{Example: } d_n(x) = \sum_k \binom{n}{k}^2 x^k.$$

We form

$$d_{n-1}(x) = \sum_k \left(\frac{n-k}{n}\right)^2 \binom{n}{k}^2 x^k,$$

$$d_{n-2}(x) = \sum_k \left(\frac{n-k}{n}\right)^2 \left(\frac{n-k-1}{n-1}\right)^2 \binom{n}{k}^2 x^k,$$

$$xd_{n-1}(x) = \sum_k \left(\frac{k}{n}\right)^2 \binom{n}{k}^2 x^k,$$

$$xd_{n-2}(x) = \sum_k \left(\frac{k(n-k)}{n(n-1)}\right)^2 \binom{n}{k}^2 x^k,$$

$$x^2d_{n-2}(x) = \sum_k \left(\frac{k(k-1)}{n(n-1)}\right)^2 \binom{n}{k}^2 x^k.$$

We try to find polynomials in $n, c_1, \ldots, c_6$ such that $c_1(n)d_n(x) + c_2(n)d_{n-1}(x) + \ldots + c_6(n)x^2d_{n-2}(x) \equiv 0$. Substituting these expressions and equating to zero the expression in front of $\binom{n}{k}^2 x^k$ yields five homogeneous equations, one for each power of k, for the six unknowns $c_1, \ldots, c_6$. Solving them yields the recurrence

$$nd_h(x) - (2n-1)(1+x)d_{n-1}(x) + (n-1)(1-x)^2d_{n-2}(x) = 0.$$

Substituting $x = 1$ and $x = -1$ yield the familiar expressions for $\Sigma \binom{n}{k}^2$ and $\Sigma (-1)^k \binom{n}{k}^2$, respectively.

THEOREM 2. Let $F_1(a,k), \ldots, F_r(a,k)$ *be as in* Theorem 1. *There is an effective algorithm to decide whether or not* $\Sigma_{k=-\infty}^{\infty} F_1(a,k) + \ldots + \Sigma_{k=-\infty} F_r(a,k) = 0$ *for all integers* $a \ge 0$.

Proof. Let $G_i(a) = \Sigma_{k=-\infty}^{\infty} F_i(a,k)$, $i = 1, \ldots, r$. By *Theorem 1* we can find linear recurrences for each of the G_is.

From Stanley (4) we can find a recurrence for $G(a) = \Sigma_{i=1}^{r} G_i(a)$. Let it be

$$P_0(a)G(a) = P_1(a)G(a-1) + \ldots + P_k(a)G(a-k), \ (a \ge k).$$

Let m be the highest integer root of $P_0(a) = 0$ (if none exists, set $m = 0$), then $G(a) = 0$ for $0 \le a \le m + k$, which implies by induction that $G(a) = 0$ for all $a \ge 0$.

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