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Solutions of Exponential Growth to Systems of Partial Differential Equations

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Integral representation formulas are established for functions of exponential growth satisfying a system of homogeneous partial differential equations with constant coefficients. This is a special case of a theorem of Ehrenpreis, but our proof is relatively short and gives the representing measures explicitly.

INTRODUCTION

Ehrenpreis [1] (Chapter VII, Th. 7.1) established an integral representation formula for functions or distributions on R^n satisfying a system of homogeneous partial differential equations with constant coefficients:

$$P_k(D)f \equiv 0, \quad k = 1,..., K, \qquad D = \left(\frac{\partial}{\partial x_1},..., \frac{\partial}{\partial x_n}\right).$$

This result is very deep and its proof rather long and intricate. In the present paper we show that if one restricts attention to C^∞ functions on R^n which are of exponential growth, it is possible to derive the formula by classical methods. Moreover, it is possible to obtain the measures featuring in the representation explicitly in terms of the represented solution. As a fringe benefit we also obtain what "Goursat data" on the boundary of the ordant $[0, \infty)^n$ (or any other ordant) is sufficient to determine the values of the solution in the interior of that ordant. Our tools are all old-fashioned: Laplace transform, Green's identity and residues.

The discrete analog of the present paper, i.e. the case where partial differential equations are replaced by partial difference equations, has been done in Zeilberger [4], which is a prerequisite. In fact, since most of the proofs are similar, we shall not give complete proofs, but instead describe how to adapt the proofs of [4] to the present situation.

We are going to prove the following

THEOREM. *Let $P_1(D),..., P_K(D)$ be a collection of partial differential operators with constant coefficents, where $D = (\partial/\partial x_1 ,..., \partial/\partial x_n)$. Then there exist algebraic*

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varieties $V_1, \ldots, V_\alpha$ *in* $\mathbb{C}^n$ *and differential operators with polynomial coefficients* $\partial_1, \ldots, \partial_\alpha$ *such that the following holds: If* $u \in C^\infty(R^n)$, $|u(x_1, \ldots, x_n)| \leqslant Ce^{A_1|x_1|+\cdots+A_n|x_n|}$, $A_i \geqslant 0$, *and*

$$P_k(D)u \equiv 0, \qquad k = 1, \ldots, K, \tag{0.1}$$

then given $A_i' > A_i$, $i = 1, \ldots, n$, *we can find the measures* $\mu_1, \ldots, \mu_\alpha$ *supported in* $V_1, \ldots, V_\alpha$ *respectively and also in* $\prod_{i=1}^n \{|\operatorname{Re} z_i| < A_i'\}$ *and a positive function* k *of polynomial growth such that*

$$u(x) = \sum_{j=1}^{\alpha} \int_{V_j} (\partial_j e^{xz}) \frac{d\mu_j}{k(z)} \quad (xz = x_1 z_1 + \cdots + x_n z_n). \tag{0.2}$$

Conversely, every function of the form (0.2) *satisfies the system* (0.1).

1. Some Preliminaries on the Laplace Transform

Given a function defined on $\{t \geqslant 0\}$ we define its Laplace transform (Williams [3], p. 1)

$$\mathscr{L}(F) = f_+(s) = \int_0^\infty e^{-st} F(t)\, dt. \tag{1.1}$$

If F is of exponential growth A:

$$|F(t)| \leqslant Ce^{At}, \qquad A \geqslant 0,$$

$f_+(s)$ is an analytic function of the complex variable s in the half plane $\operatorname{Re} s > A$. If F is C^∞ on $[0, \infty)$ then $f_+(s)$ has the asymptotic expansion ([3], p. 17)

$$f_+(s) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{s^{n+1}} \quad \operatorname{Re} s \geqslant A' > A, \tag{1.2}$$

meaning that $\lim_{s\to\infty} sf_+(s) = F(0)$, $\lim_{s\to\infty} s^2[f_+(s) - F(0)/s] = F'(0)$ etc.

We can get $F(t)$ back from $f_+(s)$ by ([3], p. 67)

$$F(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = A'} e^{st} f_+(s)\, ds \qquad (A' > A,\ t > 0). \tag{1.3}$$

Note that for $t < 0$ the right hand side vanishes. Similarly, if $F(t)$ is defined in $(-\infty, 0]$ then

$$f_-(s) = \int_{-\infty}^0 e^{-st} F(t)\, dt$$

defines an analytic function in $\operatorname{Re} s < -A$ and

$$\begin{aligned} \frac{1}{2\pi i} \int_{\operatorname{Re} s = -A'} e^{st} f_-(s)\, ds &= F(t), \qquad t < 0, \\ &= 0, \qquad t > 0. \end{aligned}$$

Thus, if $F \in C^\infty(R)$, $|F(t)| \leqslant Ce^{A|t|}$

$$F(t) = (2\pi i)^{-1} \int_{\mathrm{Re}\, s = A'} e^{st} f_+(s)\, ds + (2\pi i)^{-1} \int_{\mathrm{Re}\, s = -A'} e^{st} f_-(s)\, ds \qquad (1.4)$$

first for $t \neq 0$ and then, by continuity, for all t. With a slight change of notation, (f_+, f_-) is the Fourier-Carleman transform of F and (1.4) is the generalization of the Fourier inversion formula to functions of exponential growth.

What we did for R can be done just as well for R^n. Thus if $|F(t_1, \dots, t_n)| \leqslant \exp(A_1 | t_1 | + \cdots + A_n | t_n |)$, $R_+ = [0, \infty)$, $R_- = (-\infty, 0]$, then

$$f_{\pm \cdots \pm}(s) = \int_{R_\pm \cdots R_\pm} e^{-st} F(t)\, dt$$

$(st = s_1 t_1 + \cdots + s_n t_n$, $dt = dt_1\, dt_2 \cdots dt_n)$ is an analytic function of the complex variable $s = (s_1, \dots, s_n)$ in $\prod_{i=1}^n \pm \{\mathrm{Re}\, s_i > A_i\}$ and

$$F(t) = \frac{1}{(2\pi i)^n} \sum_{\pm} \int_{\prod_{i=1}^n \{\mathrm{Re}\, s_i = \pm A_i'\}} e^{st} f_{\pm \cdots \pm}(s)\, ds \qquad (1.5)$$

where $ds = ds_1 \cdots ds_n$, $A_i' > a_i$, $i = 1, \dots, n$ and the sum extends over all 2^n choices of signs.

Until further notice we shall consider only $F \in C^\infty(R_+{}^n)$. Analogous to (1.2) we have, if $f(s) = f_{+ \cdots +}(s) = \int_{R_+{}^n} e^{-st} F(t)\, dt$, the asymptotic expansion

$$f(s) = \sum_{\alpha \in N^n} \frac{F^{(\alpha)}(0)}{s^{\alpha+1}} \qquad \text{in} \quad \prod_{i=1}^n \{\mathrm{Re}\, s_i > A_i'\}, \qquad (1.6)$$

$A_i' > A_i$. Here, $N = \{0, 1, 2, \dots\}$ is the set of nonnegative integers and $\alpha = (\alpha_1, \dots, \alpha_n)$, $s^\alpha = s^{\alpha_n} \cdots s^{\alpha_n}$, $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$.

Let $t = (\hat{t}, \hat{\hat{t}})$ and the dual variables $s = (\hat{s}, \hat{\hat{s}})$, if we expand $f(s)$ by powers of $(\hat{s})^{-1}$, $f(s) = \sum_{\alpha' \in N^k} (a_\alpha'(\hat{\hat{s}}) / \hat{s}^{\alpha'+1})$, where k is the number of variables in $\hat{t}$, then $a_\alpha'(\hat{\hat{s}})$ is the Laplace transform (w.r.t. to $\hat{\hat{t}}$) of $\partial^{\alpha'} u$ restricted to $\hat{t} = 0$. For example, if $f(s) = \sum_{n=0}^\infty (a_n(\hat{\hat{s}}) / s_1{}^n)$, $s = (s_1, \hat{\hat{s}})$

$$a_0(\hat{\hat{s}}) = \int_{R_+^{n-1}} u(0, t_2, \dots, t_n)\, e^{-t_2 s_2 - \cdots - t_n s_n}\, dt_2 \cdots dt_n$$

$$a_1(\hat{\hat{s}}) = \int_{R_+^{n-1}} \frac{\partial u}{\partial t_1} (0, t_2, \dots, t_n)\, e^{-t_2 s_2 - \cdots - t_n s_n}\, dt_2 \cdots dt_n$$

etc. Also, the Laplace transform of $D^\beta F$ is $\mathscr{L}(D^\beta F) \equiv s^\beta \sum_{\alpha > \beta} (F^{(\alpha)}(0) / s^{\alpha+1})$.

From now on all our functions will be assumed to be of exponential growth.

2. Further Preliminaries

We already mentioned that the Laplace transform of $F \in C^\infty(R_+{}^n)$ has the asymptotic expansion

$$f(s) = \sum_{\alpha \in N^n} \frac{F^{(\alpha)}(0)}{s^{\alpha+1}}. \tag{1.6}$$

If γ is a subset of N^n we define

$$\mathscr{A}_\gamma f(s) = \sum_{\alpha \in \gamma} \frac{F^{(\alpha)}(0)}{s^{\alpha+1}}. \tag{2.1}$$

As in section 2 of [4] we define a *hyperplane* to be a subset of N^n of the form $A(r_1^{\beta_1},\ldots,r_n^{\beta_n}) = \prod_{i=1}^n \{\alpha_i \geqslant^{\beta_i} r_i\}$, where $r_i \in N$, β_i is 0 or 1 and $\geqslant^1$ is $\geqslant$, $\geqslant^0$ is $=$. The dimension of this hyperplane is $\sum_{i=1}^n \beta_i$. If $\beta_i = 1 \Rightarrow r_i = 0$, $A(r_1^{\beta_1},\ldots, r_n^{\beta_n})$ is called a *full hyperplane*. As noted is the last section, if γ is a hyperplane say $\gamma = A(r_1{}^1,\ldots, r_k{}^1; r_{k+1}^0,\ldots, r_n{}^0)$ then $\mathscr{A}_\gamma f$ is $s_1^{-r_1},\ldots, s_n^{-r_n}$ times the Laplace transform of $D_1^{r_1} \cdots D_n^{r_n} F$ restricted to $t_{k+1} = \cdots = t_n = 0$. Thus $\mathscr{A}_\gamma f$ is determined by the values of a certain partial derivative of F on a certain subset of the boundary of $R_+{}^n$. Let us also note that since

$$\{\alpha_i \geqslant r_i\} = N - \{\alpha_i = 1\} - \{\alpha_i = 2\} - \cdots - \{\alpha_i = r_i - 1\},$$

every hyperplane can be expressed in terms of full hyperplanes.

Similar to the definition in [4] which dealt with power series we define

Definition. A collection of asymptotic series $\{\phi_i\}_{i=1}^N$ is said to be polynomially dependent if there exist polynomials $\{p_i\}_{i=1}^N$ such that $\sum_{i=1}^N p_i \phi_i \equiv 0$, otherwise it will be said to be polynomially independent. $\{\phi\}_{i=1}^N$ will be said to be completely polynomially independent if $\{\mathscr{A}_\gamma \phi_i\,;\, \gamma$ a full hyperplane, $i = 1,\ldots, N\}$ is polynomially independent. If $\{\gamma_1,\ldots, \gamma_N\}$ is a collection of hyperplanes, $\phi_i = \mathscr{A}_{\gamma_i} f$ are completely polynomially independent iff $\gamma_1,\ldots, \gamma_N$ are mutually disjoint. Let $\gamma = A(r_1{}^1,\ldots, r_k{}^1; r_{i+1}^0,\ldots, r_n{}^0)$ then $\mathscr{A}_\gamma f = s_1^{-r_1-1} \cdots s_k^{-r_k-1} s_{k+1}^{-r_{k+1}-1} \cdots s_n^{-r_n-1} \phi(s_1,\ldots, s_k)$. $\phi(s_1,\ldots, s_k)$ will be termed a *part* of f. Notice that if f is holomorphic in $\prod_{i=1}^n \{\operatorname{Re} z_i > A_i'\}$ then ϕ is holomorphic in $\prod_{i=1}^k \{\operatorname{Re} z_i > A_i'\}$. As in [4] and with a similar proof we have

Lemma 2.1. *Given a collection of hyperplanes $\{\gamma\}$ and an asymptotic series f, there exists a collection of completely polynomially independent asymptotic series which are parts of f, such that each $\mathscr{A}_\gamma f$ can be expressed as a polynomial combination of asymptotic series from the second collection.*

Finally we observe that if ϕ is an asymptotic series, q a rational function and γ a full hyperplane then by (the formal) Leibnitz' rule

$$\mathscr{A}_\gamma(q\phi) = \sum q_i \mathscr{A}_{\gamma_i} \phi,$$

where q_i are rational functions and the γ_i are full hyperplanes.

3. The Greening of Laplace

Let $P(D)$ be a partial differential operator of order m and let $F \in C^\infty(R_+{}^n)$, then by Green's identity (Friedman [2], p. 3)

$$\int_{R_+{}^n} [P(D)F]e^{-st}\, dt - \int_{R_+} F(t)[P(-D)e^{-st}]\, dt = \int_{\partial R_+{}^n} B[F, e^{-st}]\, dS_x \qquad (3.1)$$

where dS_x is the surface element on $\partial R_+{}^n$ and $B[u, v] = \sum \nu_j B_j[u, v]$, ν_j being the jth cosine direction in the normal to $\partial R_+{}^n$ at $x(x \in \partial R_+{}^n)$ and $B_j[u, v]$ are bilinear expressions of u and v of order $\leqslant m - 1$. Suppose now that $P(D)F \equiv 0$ in $R_+{}^n$, then the first integral of (3.1) vanishes and since $P(-D)e^{-st} = P(s)e^{-st}$ we have

$$P(s) \int_{R_+{}^n} F(t)e^{-st}\, dt = -\int_{\partial R_+{}^n} B[F, e^{-st}]\, dS_x\,. \qquad (3.2)$$

Thus, for appropriate full hyperplanes γ we have

$$P(s) f(s) = \sum_{\gamma} p_\gamma(s)\, \mathscr{A}_\gamma f(s). \qquad (3.3)$$

Conversely if $f(s)$ satisfies (3.3) and is holomorphic in some $\prod_{i=1}^{n} \{\mathrm{Re} z_i > A_i\}$ then by using the inversion formula it is easy to see that f is the Laplace transform of a function F satisfying $P(D)F \equiv 0$.

Dividing (3.3) by $P(s)$ and employing Lemma 2.1 we get

Lemma 3.1. *Let $F \in C^\infty(R_+{}^n)$, $P(D)F \equiv 0$, and let $f(s)$ be the Laplace transform of F. Then there exist rational functions q_i and asymptotic series ϕ_i, completely polynomially independent, which are parts of f, such that*

$$f(s) = \sum_{i=1}^{n} q_i \phi_i\,,$$

Note that each ϕ_i is an asymptotic series (and a holomorphic function) depending on less that n variables.

The proof of Lemma 3.1 in [4] translates verbatim to the following

Lemma 3.2. *Let $\phi_1, \ldots, \phi_N$ be asymptotic series satisfying the relation*

$$\sum_{i=1}^{N} \sum_{l=1}^{L_i} p_{i,l} \mathscr{A}_{\gamma_l} \phi_i \equiv 0 \qquad (*)$$

where $p_{i,l}$ are polynomials and γ_l are full hyperplanes. Then there exists a collection of completely polynomially independent asymptotic series $\psi_1, \ldots, \psi_M$ such that each of the ϕ's is a rational combination of the ψ's. Furthermore, the ψ's are parts of the ϕ's.

Using (3.3) of the present paper instead of (4.3) of [4], the proof of Theorem 4.1 in [4] translates to the following

LEMMA 3.3. *Let $f(s)$ be the Laplace transform of $F \in C^\infty(R_+{}^n)$ satisfying the system*

$$P_k(D)F = 0, \qquad k = 1,\ldots, K. \tag{0.1}$$

Then there exist rational functions $\{m_j\}_1^J$ and subsets of variables $B_j \subset \{s_1, \ldots, s_n\}$ which only depend on $\{P_1, \ldots, P_K\}$ such that

$$f(s) = \sum_{j=1}^{J} m_j(s)\phi_j(B_j). \tag{3.4}$$

The $\phi_j(B_j)$'s are completely polynomially independent asymptotic series (and holomorphic functions) which are parts of f. Conversely, given asymptotic series $\phi_1, \ldots, \phi_J$, then if f, given by (3.4) is analytic in some region $\prod_{i=1}^n \{\operatorname{Re} z_i > A_i\}$, F satisfies the system (0.1).

Note that (3.4) determines F inside $R_+{}^n$ by some "Goursat data" on its boundary.

4. COMPLETION OF THE PROOF OF THE THEOREM

Lemma 5.1 of [4] translates in the present context to the following

LEMMA 4.1. *Let $p(s)$, $q(s)$ be any polynomials in n complex variables, $s = (s_1, \ldots, s_n)$ and let*

$$\Gamma = \{\operatorname{Re} s_{k+1} = \pm A'_{k+1}\} \times \cdots \times \{\operatorname{Re} s_n = \pm A'_n\}$$

$$D_\Gamma = \{-A'_{k+1} < \operatorname{Re} s_{k+1} < A'_{k+1}\} \times \cdots \times \{-A'_n < \operatorname{Re} s_n < A'_n\}$$

where $\{\operatorname{Re} s_i = \pm A_i\}$ is the infinite closed contour in the complex s_i plane consisting of the lines $\operatorname{Re} s_i = +A_i$, $\operatorname{Re} s_i = -A_i$. There exists an algebraic variety $W \subset \mathbb{C}^k$ of dimension lower than k such that if $s = (\hat{s}, \hat{\hat{s}})$, $\hat{s} = (s_1, \ldots, s_k) \in \mathbb{C}^k - W$

$$Tu(\hat{s}) \overset{\text{def}}{=} \int_\Gamma \frac{p(s)\, u(s)}{q(s)}\, ds_{k+1} \cdots ds_n \\ \sum_{p=1}^{a} \sum_{\substack{(\hat{s},\hat{\hat{s}}) \in V_p \\ \hat{\hat{s}} \in D_\Gamma}} \partial_p u(\hat{s}, \hat{\hat{s}}) \tag{4.1}$$

where $V_1, \ldots, V_a$ are some algebraic varieties and $\partial_1, \ldots, \partial_a$ are some differential operators with rational coefficients in $\partial/\partial s_{k+1} \cdots \partial/\partial s_n$. If $Tu \equiv 0$ then $\partial_p u = 0$ on V_p for every $p = 1, \ldots, a$.

We can now complete the proof of the theorem. All that was done for the ordant $R_+{}^n$ is repeated for the other $2^n - 1$ ordants $R_\pm \times \cdots \times R_\pm$, getting a representation like (3.4) for $f_{\pm \cdots \pm}(s)$ which is holomorphic in $\prod_{i=1}^n \pm \{\text{Re } s_i > A_i\}$.

Now if $B_j \subset \{s_1, ..., s_n\}$, say $B_j = \{s_1, ..., s_k\}$ then the very same $m_j(s)\,\phi_j(B_j)$ appears in the representation of $2^{n-k} f_{\pm \cdots \pm} s$ only with (possibly) a different sign. This follows from the fact that $\phi_j(B_j) = \phi_i(s_1, ..., s_k)$ corresponds to boundary information of F in $\{t_{k+1} = \cdots = t_n = 0\}$ and this set is a subset of the boundary of 2^{n-k} $R_\pm \times \cdots \times R_\pm$'s and our algorithm will yield the same $m_j(s)\,\phi_j(B_j)$ with appropriate sign in the 2^{n-k} corresponding $f_{\pm \cdots \pm}$'s. All we have to do now is to substitute the $f_{\pm \cdots \pm}$'s in (1.5) an apply Lemma 4.1 very much as it was done in section 6 of [4]. By examining our algorithm it is also seen that the last statement of the theorem holds.

5. An Example

We shall find the integral representation formula for solutions of the system

$$(D_1^2 - D_2)F \equiv 0, \qquad D_2^2 F \equiv 0 \qquad \left(D_1 = \frac{\partial}{\partial t_1}, D_2 = \frac{\partial}{\partial t_2}\right).$$

(Compare [4], section 7, example 2 and Ehrenpreis [1], p. 37).

The Green from of $D_1^2 - D_2$ is

$$B[u, v] = [vD_1 u - uD_1 v]\nu_1 - [uv]\nu_2$$

where ν_j is the jth cosine direction of the outward normal $\partial R_+{}^2$ at $x (x \in \partial R_+{}^2)$. By (3.2)

$$(s_1^2 - s_2) f(s_1, s_2) = \int_{t_1 \geqslant 0} e^{-s_2 t_2} D_1 F(0, t_2)\, dt_2 + s_1 \int_{t_1 \geqslant 0} e^{-s_2 t_2} F(0, t_2)\, dt_2 - \int_{t_2 \geqslant 0} e^{-s_1 t_1} F(t_1, 0)\, dt_1$$

i.e.

$$(s_1^2 - s_2) f(s_1, s_2) = [\text{coeff. of } s_1^{-2} \text{ in } f(s)] + s_1[\text{coeff. of } s_1^{-1} \text{ in } f(s)] - [\text{coeff. of } s_2^{-1} \text{ in } f(s)]. \tag{5.1}$$

This is an algebraic equation for asymptotic series. Instead of working with the asymptotic series $f(s) = \sum_{\alpha \in N^2} F^{(\alpha)}(0)/s^{\alpha+1}$ it would be much more convenient to work with the formal power series

$$g(\sigma_1, \sigma_2) = \sum_{\alpha \in N^2} F^{(\alpha)}(0) \sigma^\alpha.$$

Thus,

$$g(\sigma_1, \sigma_2) = \sigma_1^{-1} \sigma_2^{-1} f(\sigma_1^{-1}, \sigma_2^{-1})$$

and (5.1) turns into

$$(\sigma_2 - \sigma_1^2)\, g(\sigma_1, \sigma_2) = \sigma_1\sigma_2 \frac{\partial g}{\partial \sigma_1}(0, \sigma_2) + \sigma_2 g(0, \sigma_2) - \sigma_1^2 g(\sigma_1, 0) \quad (5.2)$$

also because $D_2^2 F \equiv 0$ we have

$$g(\sigma_1, \sigma_2) = \phi(\sigma_1) + \sigma_2\psi(\sigma_1)$$

where

$$\phi(\sigma_1) = \sum_0^\infty F^{(m,0)}(0)\sigma_1^m \quad (5.3)$$

$$\psi(\sigma_1) = \sum_0^\infty F^{(m,1)}(0)\sigma_1^m$$

Now (5.2) and (5.3) turn out to be (with $g = \tilde{f}$, $\sigma_1 = z$, $\sigma_2 = w$) the same equations as in the corresponding example 2 of [4] and by a simple algebraic manipulation which is worked out in detail there, we get

$$g(\sigma_1, \sigma_2) = a + b\sigma_1 + c(\sigma_2 + \sigma_1^2) + d(\sigma_1^3 + \sigma_1\sigma_2)$$

where $a = \phi(0) = F^{(0,0)}(0)$, $b = \phi'(0) = F^{(1,0)}(0)$, $c = \psi(0) = F^{(0,1)}(0)$, $d = \psi'(0) = F^{(1,1)}(0)$.

Thus

$$f(s_1, s_2) = as_1^{-1}s_2^{-1} + bs_1^{-2}s_2^{-1} + c(s_1^{-1}s_2^{-2} + s_1^{-3}s_2^{-1}) + d(s_1^{-4}s_2^{-1} + s_1^{-2}s_2^{-2}). \quad (5.4)$$

Until now we computed $f_{++} = f$ by applying Green's identity to $R_+ \times R_+$. Similarly we get, by applying it to the other quadrants, that

$$f_{\pm\pm}(s) = (-1)^{\pm}(-1)^{\pm} f(s). \quad (5.5)$$

Substituting (5.4), (5.5) in (1.5) and putting

$$\Gamma = \{\operatorname{Re} s_1 = \pm A_1\} \times \{\operatorname{Re} s_2 = \pm A_2\}$$

we shall obtain F as the contour integral

$$(2\pi i)^2 F(t) = a \int_\Gamma s_1^{-1}s_2^{-1}e^{st}\, ds_1\, ds_2 + b \int_\Gamma s_1^{-2}s_2^{-1}e^{st}\, ds_1\, ds_2$$
$$+ c \int_\Gamma (s_1^{-1}s_2^{-2} + s_1^{-3}s_2^{-1})e^{st}\, ds_1\, ds_2$$
$$+ d \int_\Gamma (s_1^{-4}s_2^{-1} + s_1^{-2}s_2^{-2})e^{st}\, ds_1\, ds_2 .$$

Let $E_1 = \partial/\partial s_1$, $E_2 = \partial/\partial s_2$ then

$$F(t) = ae^{st}\,|_{s=0} + bE_1e^{st}\,|_{s=0} + c(\tfrac{1}{2}E_1^2 + E_2)e^{st}\,|_{s=0} + d(\tfrac{1}{6}E_1^3 + E_1E_2)e^{st}\,|_{s=0} .$$

With the notation of the theorem we have: $V_1 = V_2 = V_3 = V_4 = \{0\}$; $\partial_1 =$ identity, $\partial_2 = E_1$, $\partial_3 = \frac{1}{2}E_1^2 + E_2$, $\partial_4 = \frac{1}{6}E_1^3 + E_1E_2$; $\mu_1 = a\delta$, $\mu_2 = b\delta$, $\mu_3 = c\delta$, $\mu_4 = d\delta$, where δ is the Dirac measure. F is completely determined by the Goursat data $F(0)$, $F^{(1,0)}(0)$, $F^{(0,1)}(0)$ and $F^{(1,1)}(0)$.

Remark. It is no accident that (5.2), (5.3) which are satisfied by g are also satisfied by $\tilde{f}$ in example 2 of [4]. For $g(\sigma) = \sum F^{(\alpha)}(0)\sigma^\alpha$ and if you put $G(\alpha) = F^{(\alpha)}(0)$ then

$$\sum a_\beta G(\alpha + \beta) = \sum a_\beta F^{(\alpha+\beta)}(0) = [D^\alpha P(D)F](0)$$

where $P(D) = \sum a_\beta D^\beta$. Thus G satisfies the partial difference equation $P(X)G \equiv 0$, where

$$X = (X_1, \ldots, X_n), (X_i f)(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n).$$

As a matter of fact, it if possible to get Lemma (3.3) directly from Theorem 4.1 of [4], however this has the disadvantage that it would only work for $F \in C^\infty(R^n)$ whereas the present proof can be easily modified so as to apply for $F \in C^m(R^n)$ as well.

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