A NEW PROOF TO EHRENPREIS'S SEMILOCAL QUOTIENT STRUCTURE THEOREM.

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Abstract. A relatively short proof of Ehrenpreis's semilocal quotient structure theorem is given. The chief tool is a representation formula for the z-transforms of solutions to systems of partial difference equations with constant coefficients.

0. Introduction. Let $R=(R_1,\ldots,R_n)$ be an n-tuple of positive numbers, and let $H(R)$ be the Banach space of bounded holomorphic functions in $D_R=\prod_{i=1}^{n}\{|z_i|<R_i\}$ with the sup norm. Let $I\subset H(R)$ be an ideal generated by (a finite number of) polynomials. If I is prime and V is the variety of common zeros of I, then Hilbert's semilocal analytic *Nullstellensatz* says that I consists of the analytic functions vanishing on V. Furthermore, there is a one-one correspondence between elements of the quotient space $\bar{u}\in H(R)/I$ and the restrictions to V, $\rho_V u$.

Ehrenpreis's semilocal quotient structure theorem (s.l. QST) is a generalization of Hilbert's semilocal analytic Nullstellensatz to ideals I which are not necessarily prime. Instead of a variety, we have the more general concept of a (polynomial) multiplicity variety (Ehrenpreis [1, p. 24]):

$$\mathfrak{B} = ((V_1,\partial_1),\ldots,(V_a,\partial_a)),$$

where $V_1,\ldots,V_a$ are (algebraic) varieties and $\partial_1,\ldots,\partial_a$ are differential operators with polynomial coefficients. The restriction operator $\rho_{\mathfrak{B}}$ is defined by $\rho_{\mathfrak{B}}G=(\partial_1 G|_{V_1},\ldots,\partial_a G|_{V_a})$. All such a-tuples form the space of analytic functions on $\mathfrak{B}$, to be denoted by $H(\mathfrak{B})$. If we restrict attention to functions G analytic in D_R, we obtain the space $H(\mathfrak{B};R)$ of analytic functions on $\mathfrak{B}\cap D_R$. Ehrenpreis's s.l. QST says roughly that for any ideal I, generated by polynomials, there exists a multiplicity variety $\mathfrak{B}$ such that $H(R)/I$ is isomorphic, via $\rho_{\mathfrak{B}}$, to $H(\mathfrak{B};R)$. In particular, $\rho_{\mathfrak{B}}u=0$ iff $u\in I$. The exact statement of the s.l. QST

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(Ehrenpreis [1, p. 74]) is very elaborate and will not be quoted here. We prefer to state it in the following dual form, which is easily seen to be equivalent.

THEOREM. *To any ideal I there corresponds a multiplicity variety* $\mathfrak{B}$ *such that if* $R'>R$, *then* $(H(R)/I)^* \subset H(\mathfrak{B};R')^*$ *and* $H(\mathfrak{B};R)^* \subset (H(R')/I)^*$.

$(H(R)/I)^*$ is the space of continuous linear functionals on $H(R)$ annihilating the ideal I. $H(\mathfrak{B};R)^*$ consists of a-tuples of measures $(d\mu_1,\dots,d\mu_a)$ supported in $(V_1,\dots,V_a)\cap D_R$ respectively. The first inclusion in the theorem means that for every $T\in(H(R)/I)^*$ we have measures $d\mu_1,\dots,d\mu_a$ such that

$$T(u) = \sum_{p=1}^{a} \int_{V_p \cap D_R'} (\partial_p u)\, d\mu_p \tag{0.1}$$

for $u\in H(R')\subset H(R)$.

The second inclusion in the theorem merely states the fact that $\rho_{\mathfrak{B}}(I)=0$.

Unlike Ehrenpreis, who starts by proving local results, we prove the semilocal theorem straight away. Our chief tool is a representation formula for the z-transforms of solutions to systems of partial difference equations with constant coefficients. The basic ideas are illustrated in Section 1, where the proof is carried out for $n=1$. Sections 2–6 contain the proof itself, and the final Section 7 illustrates the method by means of two examples.

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1. The Proof for $n=1$. The ring of polynomials in one variable is a principal ideal ring, which means that any ideal generated by polynomials is generated by a single polynomial. So $I=P(z)H(R)$ for some polynomial P.

Assume $P(z)=\prod_{i=1}^{l}(z-\alpha_i)^{s_i}$. We shall show that the required multiplicity variety is

$$\mathfrak{B} = \bigcup_{i=1}^{l} \left((\alpha_i, \text{identity}), \dots, \left(\alpha_i, \frac{d^{s_i-1}}{dz^{s_i-1}} \right) \right).$$

Let $T\in(H(R)/I)^*$; then $T(P(z)u(z))=0$ for every u and $|T(u)|\leqslant C\sup_{|z|<R}|u(z)|$. Put $f(\alpha)=T(z^\alpha), \alpha\in N$ $(N=\{0,1,2,\dots\})$. Then $|f(\alpha)|\leqslant CR^\alpha$, and $\tilde{f}(z)=\sum_{\alpha=0}^{\infty}f(\alpha)z^\alpha$ is analytic in $\{|z|<R^{-1}\}$. If $P(z)=\sum_{i=0}^{N}p_iz^i$, then for every α, $\sum_{i=0}^{N}p_if(\alpha+i)=\sum_{i=0}^{N}p_iT(z^{\alpha+1})=T(z^\alpha P(z))=0$. Thus $f:N\to\mathbb{C}$ is a

solution of the difference equation with constant coefficients

$$\sum_{i=0}^{N} p_i f(\alpha+i) \equiv 0.$$

Consider

$$P(z^{-1})\tilde{f}(z) = \sum_{i=-N}^{-1} b_i z^i + \sum_{i=0}^{\infty}\left(\sum_{i=0}^{N} p_i f(\alpha+i)\right) z^i.$$

The last term vanishes, and putting $Q(z)=z^N P(z^{-1})$, we get $\tilde{f}(z)=b(z)/Q(z)$, where b is some polynomial of degree $N-1$. Now, $\tilde{g}(z)=z^{-1}\tilde{f}(z^{-1})=c(z)/P(z)$ is a function analytic in $\{|z|>R\}$. Also, for 1 $R''>R, a_0 \in N$ $T(z^{a_0})=f(a_0)=\int_{|z|=R''}\tilde{g}(z)z^{\alpha_0}dz$, where $dz=dz/2\pi i$, since $\tilde{g}(z)=\Sigma_0 f(\alpha)z^{-\alpha-1}$. By the continuity of T for $u\in H(R')$, $R'>R$, $R<R''<R'$ we have

$$T(u) = \int_{|z|=R''} \tilde{g}(z)u(z)\,dz. \tag{1.1}$$

Expand $\tilde{g}$ in partial fractions,

$$\tilde{g}(z) = \frac{c(z)}{P(z)} = \sum_{i=1}^{l}\sum_{j=1}^{s_i} A_{i,j}(z-\alpha_i)^{-j},$$

and substitute in (1.1) to get

$$T(u) = \sum_{|\alpha_i|<R''}\sum_{j=1}^{s_i} A'_{i,j}\frac{d^{j-1}u}{dz^{j-1}}(\alpha_i),$$

which is the required (0.1) for the above $\mathfrak{B}$; $\rho_{\mathfrak{B}}(P(z)u(z))\equiv 0$ is trivial.

2. Some Preliminaries. Let $N=\{0,1,2,\dots\}$; the z-transform of a function $f: N^n\to\mathbb{C}$ is defined to be the formal power series

$$\tilde{f}(z) = \sum_{\alpha\in N^n} f(\alpha)z^\alpha \qquad (z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}).$$

Here $|f(\alpha)|\leqslant CR^\alpha$ $[R=(R_1,\dots,R_n)]$ iff $\tilde{f}$ is holomorphic in the polydisc $\prod_{i=1}^n\{|z_i|<R_i^{-1}\}$. A *hyperplane* is a subset of N^n,

$$A\left(r_1^{\beta_1},\dots,r_n^{\beta_n}\right) = \prod_{i=1}^{n}\left\{\alpha_i \overset{\beta_i}{\geqslant} r_i\right\},$$

where $r_i \in N$, β_i is 0 or 1, and $\overset{1}{\geqslant}$ is $\geqslant$, $\overset{0}{\geqslant}$ is $=$. The dimension of this hyperplane is $\sum_{i=1}^{n} \beta_i$. If $\beta_i = 1 \Rightarrow r_i = 0$, then $A(r_1^{\beta_1}, \ldots, r_n^{\beta_n})$ will be called a *full hyperplane*.

Define operators on formal power series $\mathcal{C}_\gamma \tilde{f} = \mathcal{C}(r_1^{\beta_1}, \ldots, r_n^{\beta_n})\tilde{f}(z) =$ [z-transform of f restricted to $\gamma = A(r_1^{\beta_1}, \ldots, r_n^{\beta_n})$]. Notice that for full hyperplanes γ, $\mathcal{C}_\gamma \tilde{f}$ is a differentiation with respect to a subset of the variables followed by substituting 0 for these variables, e.g.,

$$\mathcal{C}\left(0^1, \ldots, 0^1, r_{k+1}^0, \ldots, r_n^0\right)\tilde{f} = \frac{z_{k+1}^{r_{k+1}} \cdots z_n^{r_n}}{r_{k+1}! \cdots r_n!}\left[\frac{\partial^{r_{k+1} + \cdots + r_n}}{\partial z_{k+1}^{r_{k+1}} \cdots \partial z_n^{r_n}} \tilde{f}(z)\right]_{z_{k+1} = \cdots = z_n = 0}.$$

Given two power series, we define their *dotted sum*

$$\sum a_\alpha z^\alpha \dotplus \sum b_\alpha z^\alpha = \sum (a_\alpha + b_\alpha) z^\alpha - \sum_{\{\alpha;\, a_\alpha = b_\alpha\}} a_\alpha z^\alpha,$$

that is, if they have a common part it is only counted once in the addition.

Since the intersection of two full hyperplanes is another full hyperplane, we have

$$\mathcal{C}_{\gamma_1} \tilde{f} \dotplus \mathcal{C}_{\gamma_2} \tilde{f} = \mathcal{C}_{\gamma_1} \tilde{f} + \mathcal{C}_{\gamma_2} \tilde{f} - \mathcal{C}_{\gamma_1 \cap \gamma_2} \tilde{f}.$$

Similarly, if $\gamma_1, \ldots, \gamma_R$ is a collection of full hyperplanes, the dotted sum is

$$\mathcal{C}_{\gamma_1} \tilde{f} \dotplus \cdots \dotplus \mathcal{C}_{\gamma_R} \tilde{f} = \sum a_j \mathcal{C}_{\beta_j} \tilde{f},$$

where a_j are constants and β_j are full hyperplanes. For example, if $R = 3$,

$$\begin{aligned}
&\mathcal{C}_{\gamma_1} \tilde{f} \dotplus \mathcal{C}_{\gamma_2} \tilde{f} \dotplus \mathcal{C}_{\gamma_2} \tilde{f} \\
&\quad = \mathcal{C}_{\gamma_1} \tilde{f} + \mathcal{C}_{\gamma_2} \tilde{f} + \mathcal{C}_{\gamma_3} \tilde{f} \\
&\quad\quad - \mathcal{C}_{\gamma_1 \cap \gamma_2} \tilde{f} - \mathcal{C}_{\gamma_1 \cap \gamma_3} \tilde{f} - \mathcal{C}_{\gamma_2 \cap \gamma_3} \tilde{f} + \mathcal{C}_{\gamma_1 \cap \gamma_2 \cap \gamma_3} \tilde{f}.
\end{aligned}$$

We shall need the following simple combinatorial lemma.

LEMMA A. *Given a collection of hyperplanes, there exists another collection of mutually disjoint hyperplanes such that every hyperplane from the first collection can be written as a union of hyperplanes from the second collection.*

We shall prove Lemma A by induction on the maximal dimension of the hyperplanes. We need

LEMMA B. *A hyperplane minus a finite union of hyperplanes can be written as a disjoint union of hyperplanes.*

Assume

LEMMA B′. *A hyperplane minus a hyperplane can be written as a disjoint union of hyperplanes.*

Let A(k), B(k), B′(k) be the above lemmas for hyperplanes of dimension $\leqslant k$.

A(0), B(0), B′(0) are self-evident. Also B′(k)⇒B(k). Lemmas A and B will be proved if we can show

(i) B(k)⇒A(k),
(ii) [B($k-1$) and A($k-1$)]⇒B′(k).

Proof of (i). Let us call a hyperplane minus a union of hyperplanes a Z-hyperplane. By B(k), any Z-hyperplane can be written as a disjoint union of hyperplanes. Now if we have a collection of hyperplanes, they can be written as unions of mutually disjoint Z-hyperplanes; e.g., if A_1, A_2, A_3 are hyperplanes, then

$$\begin{aligned} A_1 &= (A_1 - (A_2 \cup A_3) \cap A_1) \cup (A_1 \cap A_2 - A_1 \cap A_2 \cap A_3) \\ &\quad \cup (A_1 \cap A_3 - A_1 \cap A_2 \cap A_3) \cup A_1 \cap A_2 \cap A_3, \\ A_2 &= (A_2 - (A_1 \cup A_3) \cap A_2) \cup (A_2 \cap A_1 - A_1 \cap A_2 \cap A_3) \\ &\quad \cup (A_2 \cap A_3 - A_1 \cap A_2 \cap A_3) \cup A_1 \cap A_2 \cap A_3, \end{aligned}$$

etc. So all you have to do is write each of these mutually disjoint Z-hyperplanes as a disjoint union of hyperplanes. This proves (i).

Proof of (ii). If C and D are two hyperplanes, then $C-D=C-C\cap D$. Since $C\cap D$ is a hyperplane, it suffices to consider $C-D$ where $D\subset C$. By a linear translation we can assume $C=N^k$. Thus we have to look at $N^k - A(S_1^1, \ldots, S_l^1; S_{l+1}^0, \ldots, S_k^0)$. Write this as

$$\begin{aligned} &A\big(S_1^1;\ldots;S_l^1;(S_{l+1}+1)^1;\ldots;(S_k+1)^1\big) \\ &\cup\big[\big(N^k - A\big(S_1^1;\ldots;S_l^1;(S_{l+1}+1)^1;\ldots;(S_k+1)^1\big) \\ &\qquad - A\big(S_1^1;\ldots;S_l^1;S_{l+1}^0;\ldots;S_k^0\big)\big]. \end{aligned}$$

This is a disjoint union. The first terms inside the square brackets can be written as a union of hyperplanes of dimension $\leqslant k-1$. By A($k-1$) it can be written as a disjoint union of hyperplanes. Thus the square bracket is a union of mutually disjoint Z hyperplanes of dimension $\leqslant k-1$, each of which can be written as a disjoint union of hyperplanes, by B$(k-1)$. This completes the proof of (ii).

Definition. A collection of power series $\{\phi_i\}_{i=1}^N$ is said to be *polynomially dependent* if there exist polynomials $\{p_i\}_{i=1}^N$ such that $\sum_{i=1}^N p_i\phi_i \equiv 0$; otherwise it will be said to be *polynomially independent.*

$\{\phi_i\}_{i=1}^N$ will be said to be *completely polynomially independent* if

$$\{\mathcal{Q}_\gamma \phi_i;\ \gamma \text{ a full hyperplane, } i=1,\ldots,N\}$$

is polynomially independent. If $\{\gamma_1,\ldots,\gamma_N\}$ is a collection of hyperplanes, $\phi_i = \mathcal{Q}_{\gamma_i}\tilde{f}$ are completely polynomially independent iff $\gamma_1,\ldots,\gamma_N$ are mutually disjoint.

Let $\gamma = A(r_1^1,\ldots,r_k^1; r_{k+1}^0,\ldots,r_n^0)$; then

$$\mathcal{Q}_\gamma \tilde{f} = z_1^{r_1}\cdots z_1^{r_1}\cdots z_k^{r_k} z_{k+1}^{r_{k+1}}\cdots z_n^{r_n}\phi(z_1,\ldots,z_k).$$

$\phi(z_1,\ldots,z_k)$ will be termed a *part* of $\tilde{f}$. Notice that if $\tilde{f}$ is holomorphic in $\Pi_{i=1}^n\{|z_i|<R_1^{-1}\}$, then ϕ is holomorphic in $\Pi_{i+1}^k\{|z_i|<R_i^{-1}\}$.

Corollary 2.1. *Given a collection of hyperplanes $\{\gamma\}$ and a power series $\tilde{f}$, there exists a collection of completely polynomially independent power series which are parts of $\tilde{f}$, such that each $\mathcal{Q}_\gamma \tilde{f}$ can be expressed as a polynomial combination of power series from the second collection.*

Proof. By Lemma A we can find a collection of mutually disjoint hyperplanes $\{\beta\}$ such that each γ is a union of β's. If ϕ_β is the part corresponding to $\mathcal{Q}_\beta \tilde{f}$, then $\{\phi_\beta\}$ is the required completely polynomially independent collection.

Finally, we observe that if ϕ is a power series, q a rational function and γ a full hyperplane, then by Leibnitz's rule,

$$\mathcal{Q}_\gamma(q\phi) = \sum q_i \mathcal{Q}_{\gamma_i}\phi,$$

where q_i are rational functions and γ_i are full hyperplanes.

3. An Algorithm to Solve an Extended Polynomial Relation.

A *rational function* is a quotient of polynomials; a *rational combination* of power series is a linear combination of power series with rational functions as coefficients.

LEMMA 3.1. *Let $\phi_1,\dots,\phi_N$ be power series satisfying the relation*

$$\sum_{i=1}^{N}\sum_{l=1}^{L_i} P_{i,l}\mathcal{C}_{\gamma_l}\phi_i \equiv 0, \tag{*}$$

where $p_{i,l}$ are polynomials and γ_l are full hyperplanes (and therefore $\mathcal{C}_{\gamma_l}$ is differentiation w.r.t. a certain subset of the variables followed by substituting with 0). There exists a collection of completely polynomially independent power series $\psi_1,\dots,\psi_M$ such that each of the ϕ's is a rational combination of the ψ's. Furthermore, the ψ's are parts of the ϕ's.

Proof. *Step 1:* Unless $\gamma = N^n$ (and $\mathcal{C}_\gamma$ the identity), $\mathcal{C}_\gamma\phi$ depends on fewer variables then ϕ. If we have $\mathcal{C}_\gamma\phi$'s but not ϕ itself in (*), then we can find, by Corollary 2.1, completely polynomially independent power series, which are parts of ϕ, such that each of the $\mathcal{C}_\gamma\phi$'s is a rational (in fact polynomial) combination of these. After substituting these in (*) we can assume that whenever a power series ϕ appears, we have ϕ itself and not only $\mathcal{C}_\gamma\phi$'s.

Step 2: Look at the collection of arguments of the ϕ's. Each of these is a subset of $\{z_1,\dots,z_n\}$. Let $\{B_1,\dots,B_b\}$ be a subcollection whose members contain all the sets of arguments, and let $\{B_1,\dots,B_b\}$ be minimal with respect to this property. By permuting, assume $B_1=\{z_1,\dots,z_k\}$. Expand (*) in powers of $\{z_{k+1},\dots,z_n\}$. Let $\phi_1,\dots,\phi_R$ be the ϕ's which depend on $\{z_1,\dots,z_k\}$. By equating the coefficients of $z_{k+1}^{l_{k+1}}\cdots z_n^{l_n}$ to zero we end up with a finite set of equations of the form

$$\sum_{r=1}^{R} q_r(z_1,\dots,z_k)\phi_r(z_1,\dots,z_k) = \begin{array}{l}\text{expression in power series whose}\\ \text{arguments are proper subsets of}\\ \{z_1,\dots,z_k\}.\end{array}$$

Put

$$\phi_1(z_1,\dots,z_k) = -\frac{1}{q_1}\left[\sum_{r=2}^{R} q_r\phi_r - \text{r.h.s.}\right], \tag{3.1}$$

and substitute it into the remaining equations, thus getting a system involving only $\phi_2,\dots,\phi_R$. Continue until all these equations are exhausted, not forgetting to substitute into (3.1) (and similar expressions) every result obtained subsequently. Finally, you will find that you have expressed some of the ϕ's in terms of the others and in terms of power series whose arguments are proper subsets of $\{z_1,\dots,z_k\}$. It might also happen that all $\phi_1,\dots,\phi_R$ are expressible in terms of power series with fewer variables than k. Substitute into (*), getting a new relation (*′) which has the advantage that no power series in it has $B_1=\{z_1,\dots,z_k\}$ as argument. Apply step 1 if necessary. Since the number of subsets of $\{z_1,\dots,z_n\}$ is finite $(=2^n)$, this process is bound to stop eventually, ending with the tautology $0=0$. The ψ's in terms of which the ϕ's are expressible at that stage are completely polynomially independent, since no more information can be squeezed from $0=0$.

4. Systems of Homogeneous Partial Difference Equations with Constant Coefficients. For functions $f: N^n \to \mathbb{C}$, let X_i $(i=1,\dots,n)$ be the operator defined by

$$X_i f(\alpha_1,\dots,\alpha_i,\dots,\alpha_n) = f(\alpha_1,\dots,\alpha_i+1,\dots,\alpha_n).$$

Then any polynomial P of n variables induces a partial difference equation with constant coefficients,

$$P(X)f \equiv 0, \qquad X = (X_1,\dots,X_n).$$

We are interested in studying solutions f of the homogeneous system

$$P_k(X)f \equiv 0, \qquad k = 1,\dots,K, \tag{4.1}$$

where $\{P_1,\dots,P_K\}$ is any collection of polynomials in n variables.

THEOREM 4.1. *Let $\tilde{f}(z)=\sum f(\alpha)z^\alpha$ be the z-transform of a function $f: N^n \to \mathbb{C}$ satisfying the system* (4.1). *There exist rational functions $\{m_j\}_1^J$ and subsets $B_j \subset \{z_1,\dots,z_n\}$, which depend on $\{P_1,\dots,P_K\}$ only, such that*

$$\tilde{f}(z) = \sum_{j=1}^{J} m_j(z)\phi_j(B_j). \tag{4.2}$$

The $\phi_j(B_j)$ are completely polynomially independent power series which are parts of $\tilde{f}$. Conversely, given any power series $\phi_1(B_1),\dots,\phi_J(B_J)$, then if $\tilde{f}$ is a formal power series, f is a solution to the system (4.1).

Proof. Suppose that $K=1$, i.e., we consider solutions of the single equation $P(X)f\equiv 0$. For such f,

$$P(z^{-1})\tilde{f}(z) = (\text{terms containing negative powers}) + \sum_{\alpha\in N^n}[P(X)f(\alpha)]z^\alpha.$$

Since the last term on the right hand side vanishes, we are left with the terms containing negative powers. We shall explain how to compute these.

Let $z_1^{\beta_1}\cdots z_n^{\beta_n}$ be a monomial appearing in P. The negative terms arising from $z_1^{-\beta_1}\cdots z_n^{-\beta_n}\tilde{f}(z)$ are $z_1^{-\beta_1}\cdots z_n^{-\beta_n}$[$z$-transform of f restricted to $\cup_{i=1}^n\{\alpha_i<\beta_i\}$]. (By a negative term we mean something of the form const$z_1^{\delta_1}\cdots z_n^{\delta_n}$ where at least one δ_i is negative.)

$\cup_{i=1}^n\{\alpha_i<\beta_i\}$ can be written as a union of full hyperplanes,

$$\bigcup_{i=1}^{n}\{\alpha_i<\beta_i\} = \bigcup_{m=1}^{M}\gamma_m.$$

Thus the z-transform of f restricted to $\cup_{i=1}^n\{\alpha_i<\beta_i\}$ is a dotted sum

$$\mathcal{Q}_{\gamma_1}\tilde{f}\dot{+}\mathcal{Q}_{\gamma_2}\tilde{f}\dot{+}\cdots\dot{+}\mathcal{Q}_{\gamma_M}\tilde{f},$$

which, by the remarks in Section 2, can be expressed as a linear combination of $\mathcal{Q}_\gamma\tilde{f}$'s where the γ's are full hyperplanes. Give the same treatment to all monomials appearing in P. Let z^A be the lowest possible monomial such that $z^AP(z^{-1})=Q(z)$ has only positive powers. We see that there are polynomials q_γ such that f satisfies $P(X)f\equiv 0$ iff

$$Q(z)\tilde{f}(z) = \sum q_\gamma\mathcal{Q}_\gamma\tilde{f} \tag{4.3}$$

for some collection $\{\gamma\}$ of full hyperplanes. By Corollary 2.1 there are completely polynomially independent power series $\phi_1,\ldots,\phi_J$, which are parts of $\tilde{f}$, such that each $\mathcal{Q}_\gamma\tilde{f}$ is a polynomial combination of the ϕ's. Substituting in (4.3) and dividing by Q yields

$$\tilde{f}(z) = \sum_{j=1}^{J} m_j(z)\phi_j(B_j),$$

proving the theorem for a system with one equation. Suppose we already know the theorem for the system $P_1(X)f\equiv 0,\ldots,\ P_{k-1}(X)f\equiv 0$, i.e., we have an expression

$$\tilde{f}(z) = \sum_{j=1}^{J_{k-1}} m_j^{k-1}(z)\phi_j^{k-1}\big(B_j^{k-1}\big) \tag{4.4}$$

for the general solution. Then we have that $P_k(X)f\equiv 0$ iff (4.3) holds for P_k. Substituting (4.4) in (4.3) for the P_k, we get an extended polynomial relation for the ϕ_j^{k-1}'s. Solving it by Lemma 3.1, we get a collection of completely polynomially independent power series $\phi_j^k(B_j^k)$, $j=1,\ldots,J_k$, which are parts of the ϕ_j^{k-1}'s (and by induction parts of $\tilde{f}$) such that each ϕ_j^{k-1} is some rational combination of the ϕ_i^k's. We substitute these expressions in (4.4), getting the general expression for $\tilde{f}(z)$ given f satisfying $P_1(X)f\equiv 0,\ldots,\ P_k(X)f\equiv 0$. This inductive step finishes the proof of the theorem.

5. Iterated Residues.

LEMMA 5.1. *Let $p(z)$, $q(z)$ be any polynomials in n complex variables, $z=(z_1,\ldots,z_n)$, and let*

$$\Gamma = \{|z_{k+1}|=r_{k+1}\}\times\cdots\times\{|z_n|=r_n\},$$
$$D_\Gamma = \{|z_{k+1}|<r_{k+1}\}\times\cdots\times\{|z_n|<r_n\}.$$

There exists an algebraic variety $W\subset\mathbf{C}^k$, of dimension lower then k, such that if $z=(\hat{z},\hat{\hat{z}})$, $\hat{z}=(z_1,\ldots,z_k)\in\mathbf{C}^k-W$, then

$$Tu(\hat{z}) \stackrel{\text{def}}{=} \int_\Gamma \frac{p(z)u(z)}{q(z)}\,dz_{k+1}\cdots dz_n = \sum_{p=1}^{a}\ \sum_{\substack{(\hat{z},\hat{\hat{z}})\in V_p\\ \hat{\hat{z}}\in D_\Gamma}} \partial_p u(\hat{z},\hat{\hat{z}}) \tag{5.1}$$

where $((V_1,\partial_1);\ldots;(V_a,\partial_a))$ is some multiplicity variety of dimension k and $\partial_1,\ldots,\partial_a$ are differential operators with rational coefficients in $\partial/\partial z_{k+1},\ldots,\partial/\partial z_n$. If $Tu\equiv 0$, then $\partial_p u\equiv 0$ on V_p for every $p=1,\ldots,a$.

Proof. Let us recall some elementary facts about residues (Churchill [2, pp. 155, 160]): Let C be a closed contour within and on which a function f is analytic except for a finite number of singular points $C_1,\ldots,C_m$ interior to C. If $K_1,\ldots,K_m$ denote the residues of f at these points, then $\int_C f(z)\,dz=K_1+\cdots+K_m$.

Let us also recall how to compute the residue of $f(z)=p(z)/q(z)$ at a pole z_0. If the pole is simple, the residue is $b_1=p(z_0)/q'(z_0)$. If the pole is of order m, then we look at $\phi(z)=(z-z_0))^m p(z)/q(z)$ and compute $\phi'(z_0)$. If $q(z)$ is a polynomial, we get a differential operator with rational coefficients $\mathcal{P}$ such that the residue is $\mathcal{P}(p(z))|_{z=z_0}$. We shall first prove the lemma for the case $k=n-1$, i.e., we consider the operator

$$Tu(\hat{z}) = \int_{|z_n|=r_n} \frac{p(z)u(z)}{q(z)}\,dz_n \qquad (u \text{ holomorphic in } |z_n|<r_n',\ r_n'>r_n) \tag{5.2}$$

where p, q are polynomials in n variables. The ring $C[z_1,\dots,z_n]$ of polynomials in n variables is a unique factorization domain, so we can write $q=q_1^{s_1}\cdots q_l^{s_l}$, where $q_1,\dots,q_l$ are irreducible. Also $q_2^{s_2}\cdots q_l^{s_l},\dots,q_1^{s_1}q_2^{s_2}\cdots q_l^{s_{l-1}}$ are relatively prime, so there exist polynomials $A_1,\dots,A_l$ such that

$$\frac{p(z)}{q(z)}=\frac{A_1}{q_1^{s_1}}+\cdots+\frac{A_l}{q_l^{s_l}}. \tag{5.3}$$

Consequently, by substituting (5.3) in (5.2) it is sufficient to consider operators of the form

$$Tu(\hat{z})=\int_{|z_n|=r_n}\frac{p(z)u(z)}{q^s(z)}\,dz_n, \qquad z=(\hat{z},z_n), \tag{5.4}$$

where q is irreducible. Let $q(z)=q_0(\hat{z})z_n^l+\cdots+q_l(\hat{z})$, and let $\Delta(\hat{z})$ be the discriminant; then if $\hat{z}\in\mathbb{C}^{n-1}-\Delta^{-1}(0)$, $q(z,z_n)$ has l distinct roots $\alpha_1(\hat{z}),\dots,\alpha_l(\hat{z})$. By the above discussion on computing residues we get a differential operator $\mathcal{P}$ in $\partial/\partial z_n$, with rational coefficients such that

$$Tu(\hat{z})=\sum_{\substack{q(z,z_n)=0\\ |z_n|<r_n}}\mathcal{P}u(\hat{z},z_n), \qquad \hat{z}\in\mathbb{C}^{n-1}-\Delta^{-1}(0).$$

By using (5.2) we see that if $q(z)=q_1^{s_1}\cdots q_l^{s_l}$ and $V_i=\{q_i=0\}$, there are differential operators ∂_i, with rational coefficients, such that if $W=\Delta_1^{-1}(0)\cup\cdots\cup\Delta_l^{-1}(0)$, then for $\hat{z}\in\mathbb{C}^{n-1}-W$,

$$Tu(\hat{z})=\sum_{i=1}^{l}\sum_{\substack{(\hat{z},z_n)\in V_i\\ |z_n|<r_n}}\partial_i u(z,z_n),$$

proving the lemma for $k=n-1$. If $k<n-1$, we repeat the process another $n-k-1$ times and get (5.1). The last statement of the lemma is self-evident.

6. Completion of the Proof of the s.l. QST. Let $I=\{P_1,\dots,P_K\}$. Consider $T\in(H(R)/I)^*$, and let $f(\alpha)=T(z^\alpha)$, $\alpha\in N^n$. Since $|f(\alpha)|=|T(z^\alpha)|\leqslant C\sup_{z\in D_R}|z^\alpha|=CR^\alpha$, $\tilde{f}(z)=\sum_{\alpha\in N^n}f(\alpha)z^\alpha$ is holomorphic in $\prod_{i=1}^n\{|z_i|<R_i^{-1}\}$. We have also $T(P_k(z)u(z))=0$ for $k=1,\dots,K$, $u\in H(R)$; thus $P_k(X)f(\alpha)=T(P_k(z)z^\alpha)=0$ for every $\alpha\in N^n$. It follows that f is a solution of the homogeneous system of partial difference equations $P_k(X)f\equiv 0$, $k=1,\dots,K$. By

Theorem 4.1 we have

$$\tilde{f}(z) = \sum_{j=1}^{J} m_j(z)\phi_j(B_j), \qquad B_j \subset \{z_1,\dots,z_n\}. \tag{4.2}$$

Let

$$G(z) = z^{-1}\tilde{f}(z^{-1}) = \sum_{\alpha \in N^n} f(\alpha) z^{-(\alpha+1)},$$

where $z^{-1} = z_1^{-1} \cdots z_n^{-1}$ or $(z_1^{-1},\dots,z_n^{-1})$ according to context and $-(\alpha+1) = (-(\alpha_1+1),\dots,-(\alpha_n+1))$. $G(z)$ is holomorphic in $\prod_{i=1}^{N}\{|z_i| > R_i\}$, and for $u \in H(R')$, $R < R'' < R'$, $\Gamma_{R''} = \prod_{i=1}^{n}\{|z_i| = R_i''\}$, we have

$$T(u) = \int_{\Gamma_{R''}} G(z)u(z)\,dz_1 \cdots dz_n.$$

To verify this formula, first put $u = z^m$, for which it is merely Cauchy's formula, and then extend to all $u \in H(R')$ by the continuity of T on $H(R)$. To avoid constants we put $dz_i = dz_i/2\pi i$. Now let $n_j(z) = z^{-1}m_j(z^{-1})$, $\psi_j(B_j) = \phi_j(B_j^{-1})$ (where if $B_j = \{z_{i_1},\dots,z_{i_k}\}$, then $B_j^{-1} = \{z_{i_1}^{-1},\dots,z_{i_k}^{-1}\}$). Then n_j are rational functions, ψ_j are holomorphic in $\prod_{i=1}^{n}\{|z_i| > R_i\}$, and $G(z) = \sum_{j=1}^{J} n_j(z)\psi_j(B_j)$. Thus

$$T(u) = \sum_{j=1}^{J} \int_{\Gamma_{R''}} \psi_j(B_j) n_j(z) u(z)\,dz = \sum_{j=1}^{J} \int_{\Gamma_j} \psi_j(B_j) T_j(u)\,dB_j, \tag{6.1}$$

where Γ_j is the polycircle corresponding to the variables B_j, and dB_j is the corresponding differential. For example, if $B_j = \{z_1,\dots,z_k\}$, then $\Gamma_j = \{|z_1| = R_1''\} \times \cdots \times \{|z_k| = R_k''\}$, $dB_j = dz_1 \cdots dz_k$. If Γ_j' is the polycircle corresponding to the complement of B_j, B_j', then

$$T_j(u) = \int_{\Gamma_j'} n_j(z)u(z)\,dB_j.$$

By Lemma 5.1 there corresponds to T_j a multiplicity variety $\mathfrak{B}_j$. The required multiplicity variety is $\mathfrak{B} = \mathfrak{B}_1 \cup \cdots \cup \mathfrak{B}_J$. Indeed, the representation formula (0.1) is obtained by substituting the expressions (5.1) for each T_j in (6.1). The exceptional sets W_j need not bother us, since $W_j \cap \Gamma_j$ has measure zero with respect to dB_j. Although the differential operators in $\mathfrak{B}$ have rational coefficients, we can take a common denominator which can be absorbed in the measures, and we finally get a multiplicity variety $\mathfrak{B}$ with polynomial coefficients.

It only remains to check that $\rho(u)\equiv 0$ for $u\in I$. By the last sentence of Lemma 4.1, $\tilde{f}_j(z)=m_j(z)\phi_j(B_j)$ represents the z-transform of f_j, which is a solution of the system $P_1(X)f\equiv\cdots\equiv P_K(X)f\equiv 0$. Thus for every $u\in I$, $T_j u=0$. By the last sentence of Lemma 5.1, $\rho_{\mathfrak{B}_j}u\equiv 0$ and thus $\rho_{\mathfrak{B}}u\equiv 0$.

7. Two Examples. The method of this paper will be now illustrated by means of two examples.

Example 1. Compute the multiplicity variety in $\mathbb{C}^3$ corresponding to the ideal I generated by $\{z_2^2, z_3^2, z_2-z_1z_3\}$ (Björk [3, p. 22]).

If $X_2^2f\equiv 0$, $X_3^2f\equiv 0$, then $f(m,n,k)=0$ for $n\geqslant 2$ or $k\geqslant 2$, and

$$\tilde{f}(z_1,z_2,z_3) = \phi_{00}(z_1)+z_2\phi_{10}(z_1)+z_3\phi_{01}(z_1)+z_2z_3\phi_{11}(z_1). \tag{7.1}$$

Now $(X_2-X_1X_3)f\equiv 0$ iff

$$\begin{aligned}(z_2^{-1}-z_1^{-1}z_3^{-1})\tilde{f} = {}& z_2^{-1}\tilde{f}(z_1,0,z_3)\\ &-z_1^{-1}z_3^{-1}\big[\tilde{f}(0,z_2,z_3)+\tilde{f}(z_1,z_2,0)-\tilde{f}(0,z_2,0)\big].\end{aligned}$$

Multiply both sides by $z_1z_2z_3$:

$$(z_1z_3-z_2)\tilde{f} = z_1z_3\tilde{f}(z_1,0,z_3) - z_2\big[\tilde{f}(0,z_2,z_3)+\tilde{f}(z_1,z_2,0)-\tilde{f}(0,z_2,0)\big].$$

Substitute (7.1):

$$\begin{aligned}&(z_1z_3-z_2)\big[\phi_{00}(z_1)+z_2\phi_{10}(z_1)+z_3\phi_{01}(z_1)+z_2z_3\phi_{11}(z_1)\big]\\ &\quad= z_1z_3\big[\phi_{00}(z_1)+z_3\phi_{01}(z_1)\big]\\ &\quad\quad - z_2\big[z_3\phi_{01}(0)+z_2z_3\phi_{11}(0)+\phi_{00}(z_1)+z_2\phi_{10}(z_1)\big].\end{aligned}$$

Rearranging

$$\big[z_1\phi_{10}(z_1)-\phi_{01}(z_1)-\phi_{01}(0)\big]-z_2\big[\phi_{11}(z_1)+\phi_{11}(0)\big]+z_3\big[z_1\phi_{11}(z_1)\big]\equiv 0,$$

we get the equations

$$\begin{aligned}z_1\phi_{10}(z_1)-\phi_{01}(z_1)-\phi_{01}(0)&\equiv 0,\\ \phi_{11}(z_1)&\equiv 0.\end{aligned}$$

Thus $\phi_{01}(z_1)=z_1\phi_{10}(z_1)$, and substituting into (7.1) we get

$$\begin{aligned}\tilde{f}(z_1,z_2,z_3) &= \phi_{00}(z_1)+z_2\phi_{10}(z_1)+z_3z_1\phi_{10}(z_1)\\ &= \phi_{00}(z_1)+(z_2+z_1z_3)\phi_{10}(z_1).\end{aligned}$$

Now

$$G(z)=\frac{1}{z_1z_2z_3}\tilde{f}\left(z_1^{-1},z_2^{-1},z_3^{-1}\right)=\frac{\psi_{00}(z_1)}{z_1z_2z_3}+\frac{(z_2+z_1z_3)}{(z_1z_2z_3)^2}\psi_{10}(z_1).$$

Hence if $T(I)=0$,

$$\begin{aligned}T(u) &= \int_{\Gamma_R} G(z)u(z)\,dz\\ &= \int_{|z_1|=r_1}\frac{\psi_{00}(z_1)}{z_1}\int\frac{u}{z_2z_3}\,dz_2\,dz_3\\ &\quad+\int_{|z_1|=r_1}\frac{\psi_{10}(z_1)}{z_1^2}\int\left(z_2^{-1}z_3^{-2}+z_1z_2^{-2}z_3^{-1}\right)u(z_1,z_2,z_3)\,dz_2\,dz_3.\end{aligned}$$

So

$T(u)=$

$$\int_{|z_1|=r_1}\frac{\psi_{00}(z_1)}{z_1}u(z_1,0,0)\,dz_1+\int\frac{\psi_{10}(z_1)}{z_1^2}\left[\left(\frac{\partial}{\partial z_3}+z_1\frac{\partial}{\partial z_2}\right)u\right]_{z_2=z_3=0}dz_1,$$

or

$$T(u)=\int_{z_2=z_3=0}u\,d\mu_1+\int_{z_2=z_3=0}\left[z_1\frac{\partial u}{\partial z_2}+\frac{\partial u}{\partial z_3}\right]d\mu_2,$$

and the multiplicity variety corresponding to I is

$$\left((z_2=z_3=0;\ \text{identity}),\left(z_2=z_3=0;\ z_1\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_3}\right)\right).$$

Example 2 (Ehrenpreis [1, p. 37]). Compute the multiplicity variety in $\mathbb{C}^2$ corresponding to the ideal generated by $\{z^2-w,w^2\}$.

$X_2^2f\equiv 0$ iff

$$\tilde{f}(z,w)=\phi(z)+w\psi(z). \tag{7.2}$$

$(X_1^2-X_2)f\equiv 0$ iff

$$(z^{-2}-w^{-1})\tilde{f}(z,w)=z^{-2}\tilde{f}(0,w)+z^{-1}\frac{\partial\tilde{f}}{\partial z}(0,w)-w^{-1}\tilde{f}(z,0).$$

Multiply by z^2w:

$$(w-z^2)\tilde{f}(z,w) = w\tilde{f}(0,w) + zw\frac{\partial f}{\partial z}(0,w) - z^2\tilde{f}(z,0).$$

Substitute (7.2):

$$(w-z^2)[\phi(z)+w\psi(z)] = w[\phi(0)+w\psi(0)] + zw[\phi'(0)+w\psi'(0)] - z^2\phi(z).$$

Rearranging and dividing by w,

$$\phi(z) + (w-z^2)\psi(z) - \phi(0) - w\psi(0) - z\phi'(0) - zw\psi'(0) \equiv 0.$$

Comparing coefficients of w, we get

$$\begin{aligned} \phi(z) - z^2\psi(z) - \phi(0) - z\phi'(0) &= 0, \\ \psi(z) - \psi(0) - z\psi'(0) &= 0. \end{aligned}$$

Let $\psi(0)=a$, $\psi'(0)=b$, $\phi(0)=c$, $\phi'(0)=d$; then

$$\psi(z) = a + bz, \qquad \phi(z) = bz^3 + az^2 + dz + c.$$

Substituting these expressions in (7.2), we get

$$\begin{aligned} \tilde{f}(z,w) &= bz^3 + az^2 + dz + c + aw + bzw = a(w+z^2) + b(z^3+zw) + c + dz, \\ G(z,w) &= z^{-1}w^{-1}\tilde{f}(z^{-1},w^{-1}) = a(z^{-1}w^{-2}+z^{-3}w^{-1}) + b(z^{-4}w^{-1}+z^{-2}w^{-2}) \\ &\quad + cz^{-1}w^{-1} + dz^{-2}w^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} T(u) &= a\int\int(z^{-1}w^{-2}+z^{-3}w^{-1})u(z,w)\,dz\,dw \\ &\quad + b\int\int(z^{-4}w^{-1}+z^{-2}w^{-2})u(z,w)\,dz\,dw \\ &\quad + c\int\int z^{-1}w^{-1}u(z,w)\,dz\,dw + d\int\int z^{-2}w^{-1}u(z,w)\,dz\,dw \\ &= a\left[\frac{1}{2}\frac{\partial u}{\partial z^2} + \frac{\partial u}{\partial w}\right]_{z=w=0} + b\left[\frac{1}{6}\frac{\partial u}{\partial z^3} + \frac{\partial^2 u}{\partial z\,\partial w}\right]_{z=w=0} \\ &\quad + cu(0,0) + d\frac{\partial u}{\partial z}(0,0). \end{aligned}$$

It turns out that the multiplicity variety is

$$((0,0),\text{ identity}); \quad \left((0,0), \frac{\partial}{\partial z}\right)$$

$$\left((0,0), \frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial w}\right); \quad \left((0,0), \frac{1}{6}\frac{\partial^3}{\partial z^3} + \frac{\partial^2}{\partial z\,\partial w}\right).$$

Finally, let us remark that our Theorem 4.1 similarly yields Ehrenpreis's Theorem 3.2 [1, p. 75]. Furthermore, the methods of this paper easily generalize to submodules of $H(R)^l$ which are generated by vectors of polynomials, yielding vector multiplicity varieties (Ehrenpreis [1, Theorem 3.3, p. 76]).

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