JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 58, 405–418 (1977)

Further Properties of Discrete Analytic Functions

DORON ZEILBERGER AND HARRY DYM

Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel

Submitted by R. P. Boas

Analogs of the classical theorems of Liouville, Phragmén–Lindelöf, and Paley–Wiener are proved in the class of discrete analytic functions.

1. INTRODUCTION

Let Z be the group of integers, let $Z_h = hZ = \{hm; m \in Z\}$ for $h > 0$, and consider the class of functions $F\colon Z_h \times Z_h \to \mathbb{C}$ such that

$$\frac{F(x+h, y+h) - F(x, y)}{h(1+i)} = \frac{F(x, y+h) - F(x+h, y)}{h(i-1)} \tag{1.1}$$

for every point $(x, y) \in Z_h \times Z_h$.

If $Z_h \times Z_h$ is identified with the lattice $\{h(m + in); m, n \text{ integers}\}$ embedded in the complex plane, then condition (1.1) is seen to be an "analyticity" condition: On each unit square of the lattice, the difference quotients along the two diagonals are the same.

The above definition of analyticity was introduced by Ferrand [1] and many properties of such functions were found by Duffin [2]. In this paper, discrete analytic functions are further investigated and discrete analogs of the classical theorems of Liouville, Phragmén–Lindelöf, and Paley–Wiener are proved.

If condition (1.1) holds for a particular point $(x_0, y_0) \in Z_h \times Z_h$ we say that F is discrete analytic in the unit square $\{(x_0, y_0), (x_0 + h, y_0), (x_0, y_0 + h), (x_0 + h, y_0 + h)\}$. Following Duffin [2] we define a region in $Z_h \times Z_h$ as a union of unit squares and say that F is discrete analytic in the region if it is discrete analytic in each of its unit squares.

In the following (x, y) and $x + iy$ will be used interchangeably to denote a point in $Z_h \times Z_h$; also notice that condition (1.1) is equivalent to

$$F(x, y) + iF(x+h, y) - F(x+h, y+h) - iF(x, y+h) = 0.$$

ISSN 0022-247X

In particular, for the lattice $Z \times Z$, F is discrete analytic in

$$\{(m, n), (m+1, n), (m+1, n+1), (m, n+1)\}$$

if

$$F(m, n) + iF(m+1, n) - F(m+1, n+1) - iF(m, n+1) = 0. \tag{1.2}$$

2. Discrete Analytic Functions of Polynomial Growth

Duffin [2] defined a biopolynomial to be a discrete analytic function which assumes the values of one polynomial on the even[1] lattice points and the values of another (possibly the same) polynomial on the odd lattice points.

Theorem 1. *Every discrete analytic function F of polynomial growth is a bipolynomial.*

Proof. Assume $h = 1$ (the proof for general h is similar) and let $F(m, n)$ be a discrete analytic function of polynomial growth: $|F(m, n)| \leqslant C(|m| + |n|)^k$, for some constants C and k. Then [6, Chapter 12] F is the Fourier transform of a distribution D on the two-dimensional torus T^2 $(=\hat{Z}^2)$,

$$F(m, n) = D(e^{imt+ins}).$$

Substituting this into (1.2) one gets

$$\begin{aligned} 0 &= F(m, n) + iF(m+1, n) - F(m+1, n+1) - iF(m, n+1) \\ &= D(e^{imt+ins}) + iD(e^{i(m+1)t+ins}) - D(e^{i(m+1)t+i(n+1)s}) - iD(e^{imt+i(n+1)s}) \\ &= D(e^{imt+ins} + ie^{i(m+1)t+ins} - e^{i(m+1)t+i(n+1)s} - ie^{imt+i(n+1)s}) \\ &= D((1 + ie^{it} - e^{it+is} - ie^{is})\, e^{imt+ins}) = 0 \end{aligned}$$

for every point $(m, n) \in Z^2$. Thus

$$(1 + ie^{it} - e^{it+is} - ie^{is})\, D \equiv 0.$$

The only roots of $1 + ie^{it} - e^{it+is} - ie^{is} = 0$ are the points $(0, 0)$ and (π, π), which implies that D is supported in these points. So if δ denotes the Dirac measure and $\delta_{(\pi,\pi)}$ denotes the Dirac measure translated by (π, π), D can be written [3, p. 103] as a finite sum of derivatives of δ and $\delta_{(\pi,\pi)}$:

$$D = \sum_{\substack{k=0\\l=0}}^{K,L} a_{kl} \frac{\partial^{k+l}}{\partial^k\, \partial^l} \delta + \sum_{\substack{k=0\\l=0}}^{K,L} b_{kl} \frac{\partial^{k+l}}{\partial^k\, \partial^l} \delta_{(\pi,\pi)} .$$

[1] The lattice point (mh, nh) is said to be even (odd) if $m + n$ is even (odd).

So

$$F(m, n) = D(e^{imt+ins})$$
$$= \sum_{\substack{k=0\\l=0}}^{K,L} a_{kl}(-1)^{k+l}(im)^k(in)^l + \sum_{\substack{k=0\\l=0}}^{K,L} b_{kl}(-1)^{k+l}(im)^k(in)^l e^{im\pi+in\pi}$$

$$P(m, n) + (-1)^{m+n} Q(m, n) = P(m, n) + Q(m, n), \qquad m + n \text{ even},$$
$$= P(m, n) - Q(m, n), \qquad m + n \text{ odd},$$

where P, Q are the polynomials

$$P(m, n) = \sum_{\substack{k=0\\l=0}}^{K,L} (-i)^{k+l} a_{kl} m^k n^l,$$

$$Q(m, n) = \sum_{\substack{k=0\\l=0}}^{K,L} (-i)^{k+l} b_{kl} m^k n^l.$$

In the algebra $C^\infty(T^2)$ the discrete analytic functions of polynomial growth are exactly the Fourier transforms of distributions which annihilate the ideal $(1 + ie^{it} - e^{it+is} - ie^{is})\, C^\infty(T^2)$. If the mesh size of the lattice is h instead of 1, then the discrete analytic functions of polynomial growth are the Fourier transforms of distributions on $(T/h) \times (T/h)$ which annihilate the ideal

$$a_h(t, s)\, C^\infty\left(\frac{T}{h} \times \frac{T}{h}\right),$$

where

$$a_h(t, s) = \frac{1 + ie^{iht} - e^{iht+ihs} - ie^{ihs}}{-(1+i)\,h}.$$

Now

$$-(1 + i)\,ha_h(t, s) = 1 + ie^{iht} - e^{iht+ihs} - ie^{ihs} = 1 + i(1 + iht + O(h^2)),$$
$$-(1 + iht + ihs + O(h^2)) - i(1 + ihs + O(h^2)) = -(1 + i)\,h[(t + is) + O(h)].$$

So $a_h(t, s) = (t + is) + O(h)$.

Now let $f(z) = f(x + iy)$ be a (continuous) entire function of polynomial growth. Then $f(x, y) = \hat{D}$ for some temperate distribution[2] D, and by the Cauchy–Riemann equation

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\hat{D} = 0$$

[2] A temperate distribution is a continuous linear functional on the Fréchet space $C_\downarrow^\infty(R^2)$, the space of rapidly decreasing functions, cf. [3, p. 134].

or, via the Fourier transform

$$(t + is)\, D \equiv 0.$$

So (continuous) entire functions of polynomial growth are exactly the Fourier transforms of temperate distributions which annihilate the ideal $(t + is)\, C_{\downarrow}^{\infty}(R \times R) = a_0(t, s)\, C_{\downarrow}^{\infty}(R \times R)$. But $(t + is)\, D \equiv 0$ implies that D is supported at the point (0, 0) and therefore is a finite sum of derivatives of the Dirac measure δ, and the familiar Liouville theorem drops out: An entire function of polynomial growth is a polynomial. Since $a_0(t, s) = t + is$ vanishes just at one point (namely, $(0, 0) \in R^2$) while $a_h(t, s)$ vanishes at two points ((0, 0) and $(\pi/h, \pi/h) \in (T/h) \times (T/h)$) it is clear why, in the discrete theory, we e·counter bipolynomials and not just polynomials.

3. A Phragmén–Lindelöf Principle for Discrete Analytic Functions

In the classical theory of analytic functions there are a number of theorems, associated with the names of Phragmén and Lindelöf, which compare the growth of an analytic function inside a sector, or a strip, with the growth of the function on the boundary. The only sectors that can be treated conveniently in the discrete theory are, evidently, the ones bounded by the axes. For the sake of definiteness we choose to consider $Z^+ \times Z^+ = \{(m, n);\ m \text{ and } n \text{ integers},\ m \geqslant 0,\ n \geqslant 0\}$.

THEOREM 2. *Let $F(m, n)$ be a discrete analytic function in the quarter lattice $Z^+ \times Z^+$, and assume that there are constants $T > 1$, $S > 1$, and C_1, C_2 such that*

$$|F(m, 0)| \leqslant C_1 T^m, \qquad m \geqslant 0 \tag{3.1a}$$

$$|F(0, n)| \leqslant C_2 S^n, \qquad n \geqslant 0. \tag{3.1b}$$

Then for every $T_1 > T$, $S_1 > \max\{S, (T + 1)/(T - 1)\}$ there exists a constant C such that $|F(m, n)| \leqslant C T_1^m S_1^n$ $\forall (m, n) \in Z^+ \times Z^+$.

Proof. Consider the formal power series

$$\mathbf{F}(z, w) = \sum_{\substack{m=0\\ n=0}}^{\infty} F(m, n)\, z^m w^n. \tag{3.2}$$

Then

$$(1 + iz - zw - iw)\, \mathbf{F}(z, w) = (1 + iz)\, \phi_F(z) + (1 - iw)\, \psi_F(w) - F(0, 0)$$
$$- \sum_{\substack{m=0\\ n=0}}^{\infty} [F(m, n) + iF(m + 1, n) - F(m + 1, n + 1) - iF(m, n + 1)]\, z^{m+1} w^{n+1} \tag{3.3}$$

where ϕ_F, ψ_F are the formal power series

$$\phi_F(z) = \sum_{m=0}^{\infty} F(m, 0)\, z^m,$$

$$\psi_F(w) = \sum_{n=0}^{\infty} F(0, n)\, w^n.$$

Now, since $F(m, n)$ is discrete analytic in $Z^+ \times Z^+$, the last term on the rhs of (3.3) is zero and consequently

$$(1 + iz - zw - iw)\,\mathbf{F}(z, w) = (1 + iz)\phi_F(z) + (1 + iw)\psi_F(w) - F(0, 0). \quad (3.4)$$

Until now, ϕ_F, ψ_F and $\mathbf{F}$ were considered as formal power series, but by (3.1a), $\phi_F(z)$ is convergent in $\{|z| < 1/T\}$ and represents an analytic function there. Similarly, by (3.1b) $\psi_F(w)$ represents an analytic function in $\{|w| < 1/S\}$ so the rhs of (3.4) is an analytic function of two complex variables in the polydisc $\{|z| < 1/T\} \times \{|w| < 1/S\}$. Thus

$$\mathbf{F}(z, w) = \frac{(1 + iz)\phi_F(z) + (1 - iw)\psi_F(w) - F(0, 0)}{1 + iz - zw - iw},$$

which was only defined a priori as formal power series, is a convergent power series in the polydisc $\{|z| < 1/T\} \times \{|w| < 1/S'\}$ where $S' = \max\{S, (T + 1)/(T - 1)\}$.

Finally, since $F(m, n)$ is the coefficient of $z^m w^n$ in the Taylor expansion of $\mathbf{F}(z, w)$ it follows by Cauchy's inequality that for every $T_1 > T$, $S_1 > \max\{S, (T + 1)/(T - 1)\}$ there exists a constant C such that

$$|F(m, n)| \leqslant C T_1{}^m S_1{}^n$$

for every point $(m, n) \in Z^+ \times Z^+$.

4. Discrete Fourier Analysis and Discrete Analytic Functions

The characters of the group R can be identified as the class of functions e^{isx}: $s \in R = \hat{R}$. Clearly each character can be extended analytically to the whole complex plane as $e^{is(x+iy)} = e^{isx}e^{-sy}$. Now look at the group Z, with characters $e^{imt}(t \in T = \hat{Z})$. One may ask: What is the natural discrete analytic extension of e^{imt} $(m \in Z)$ to the whole discrete lattice Z^2? With the continuous example in mind, let us try for an extension of the form

$$e^{imt} \cdot \phi_t(n) \qquad \text{with} \qquad \phi_t(0) = 1.$$

Substituting this into (1.2) we obtain $e^{imt}[\phi_t(n) + ie^{it}\phi_t(n) - e^{it}\phi_t(n+1) - i\phi_t(n+1)] = 0$. Therefore

$$(1 + ie^{it})\,\phi_t(n) = (i + e^{it})\,\phi_t(n+1).$$

If $t \neq \pm(\pi/2)$ one gets

$$\phi_t(n) = \left(\frac{1 + ie^{it}}{e^{it} + i}\right)^n \qquad n \in Z.$$

So the natural analog to the exponential function $e^{isz}, e^{isz} = e^{isx}e^{-sy}$ $(s \in R)$ is

$$\tilde{e}(it; m + in) = e^{imt}\left(\frac{1 + ie^{it}}{i + e^{it}}\right)^n \qquad (t \in T,\, t \neq \pm\pi/2) \tag{4.1}$$

which we shall call the discrete exponential function. This coincides with the discrete exponential function introduced by Ferrand [1]:

$$e(m + in; s) = \left(\frac{2 + s}{2 - s}\right)^m \left(\frac{2 + is}{2 - is}\right)^n$$

if $(2 + s)/(2 - s) = e^{it}$. As the above considerations showed, our exponential function seems to be a more natural analog of the continuous exponential function, at least for the purpose of doing Fourier analysis. In fact, the main theme of this paper is that continuous analytic function theory (on $R^2 = \mathbb{C}$) is what it is because of the dual group of R: $\hat{R} = R$, and discrete analytic function theory on $Z_h{}^2$ is what it is because of the dual group of Z_h: $\hat{Z}_h = T/h$. Notice that if $t = \pi/2$ $[= -(\pi/2)]$, then (4.1) still defines a meaningful exponential function for $n \geqslant 0$ $[n \leqslant 0]$. The analog on the lattice $Z_h \times Z_h$ is

$$\tilde{e}_h(it; mh + inh) = e^{imth}\left(\frac{1 + ie^{ith}}{e^{ith} + i}\right)^n.$$

Notice that for any fixed $t \in R$

$$\tilde{e}_h(it; x + iy) = e^{itx}\left[\left(\frac{1 + ie^{ith}}{e^{ith} + i}\right)^{1/h}\right]^y \to e^{it(x+iy)}$$

as $h \downarrow 0$, since

$$\left(\frac{1 + ie^{ith}}{e^{ith} + i}\right)^{1/h} \to e^{-t}.$$

Let us return to the case $h = 1$. Immitating the notation in the continuous case we let Z^{2+} denote the upper half-lattice $\{(m, n); n \geqslant 0\}$ and define the class $H^{2+}(Z)$ as follows:

DEFINITION. A discrete analytic function F on Z^{2+} is said to belong to $H^{2+}(Z)$ if

$$\sup_{n\geqslant 0}\left(\sum_{m=-\infty}^{\infty}|F(m+in)|^2\right)^{1/2}<\infty. \tag{4.2}$$

We are now in a position to give a discrete analog to the famed one-sided Paley–Wiener theorem.

THEOREM W3. *If F is discrete analytic on Z^{2+} and if $\sum_{-\infty}^{\infty}|F(m)|^2<\infty$ then $F\in H^{2+}(Z)$ iff*

$$F_0{}^v(t)=\sum_{-\infty}^{\infty}F(m)\,e^{-imt}=0 \qquad \text{a.e.}\quad in \quad (-\pi,0)$$

and in that case F has the representation

$$F(m+in)=(1/2\pi)\int_0^{\pi}\tilde{e}(it;m+in)\,F_0{}^v(t)\,dt. \tag{4.3}$$

Proof. Suppose $F_0{}^v(t)=0$ a.e. in $(-\pi,0)$ then (4.3) defines a discrete analytic extension of the starting sequence $F(m)$ to the upper half-lattice Z^{2+}; and

$$F_n(m)=F(m+in)=\frac{1}{2\pi}\int_0^{\pi}F_0{}^v(t)\left(\frac{1+ie^{it}}{i+e^{it}}\right)^n e^{imt}\,dt$$

implies, by Plancherel, that for $n\geqslant 0$

$$\sum_{m=-\infty}^{\infty}|F(m+in)|^2=\|F_n\|_{l^2}^2$$

$$=\frac{1}{2\pi}\int_0^{\pi}\left|\frac{1+ie^{it}}{i+e^{it}}\right|^{2n}|F_0{}^v(t)|^2\,dt\leqslant\frac{1}{2\pi}\int_0^{\pi}F_0{}^v(t)^2\,dt.$$

Since $|(1+ie^{it})/(i+e^{it})|\leqslant 1$ for $t\in[0,\pi]$. This proves $F\in H^{2+}(Z)$.

Conversely, if $F\in H^{2+}(Z)$ then $F_n\in l^2=L^2(Z)$ for $n\geqslant 0$ and

$$F_n{}^v(t)=\sum_{m=-\infty}^{\infty}F(m+in)\,e^{-imt}\in L^2(T).$$

Now, using (1.2) it is readily seen that

$$(1+ie^{it})\,F_n{}^v(t)=(i+e^{it})\,F_{n+1}^{v}(t) \qquad (n\geqslant 0).$$

Since the Fourier coefficients of the two sides match:

$$\begin{aligned}(1/2\pi)\int (1+ie^{it})F_n{}^v(t)\,e^{imt}\,dt\\ &= F_n(m)+iF_n(m+1)\\ &= F(m+in)+iF(m+1+in) = iF(m+i(n+1))+F(m+1+i(n+1))\\ &= iF_{n+1}(m)+F_{n+1}(m+1) = (1/2\pi)\int (i+e^{it})F_{n+1}^v(t)\,e^{imt}\,dt.\end{aligned}$$

Therefore

$$F_n{}^v(t) = \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n F_0{}^v(t) \qquad (n \geqslant 0). \tag{4.4}$$

Now, suppose that $F_0{}^v$ does not vanish a.e. in $(-\pi, 0]$. Then there exists an interval $[\alpha, \beta] \subset (-\pi, 0)$ such that $\int_\alpha^\beta |F_0{}^v|^2 \neq 0$, and so

$$\begin{aligned}\sum |F(m+in)|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_n{}^v(t)|^2\,dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{1+ie^{it}}{i+e^{it}}\right)^n F_0{}^v(t)\right|^2 dt \geqslant k^{2n}\frac{1}{2\pi}\int_\alpha^\beta |F_0{}^v(t)|^2\,dt\end{aligned}$$

in which

$$k = \min\left(\left|\frac{1+ie^{it}}{i+e^{it}}\right|;\ \alpha \leqslant t \leqslant \beta\right) > 1$$

Hence $F_0{}^v(t) = \sum_{m=-\infty}^{\infty} F(m)\,e^{-imt} = 0$ a.e. in $(-\pi, 0]$. By (4.4), for $n \geqslant 0$

$$\begin{aligned}F(m+in) = F_n(m) &= \frac{1}{2\pi}\int_0^\pi e^{imt}\left(\frac{1+ie^{it}}{i+e^{it}}\right)^n F_0{}^v(t)\,dt\\ &= \frac{1}{2\pi}\int_0^\pi \tilde{e}(it; m+in)\,F_0{}^v(t)\,dt,\end{aligned}$$

establishing (4.3).

By (4.3) a function $F(m+in)$ of class $H^{2+}(Z)$ is uniquely determined by its restriction to the discrete real line $n = 0$, so $H^{2+}(Z)$ can be viewed, in an obvious fashion, as a subset of $l^2 = L^2(Z)$ and Theorem 3 tells us that

$$L^2(Z) = L^2(-\pi,\pi)^\wedge \supset L^2(0,\pi)^\wedge = H^{2+}(Z)$$

which is in perfect analogy with the line (cf. [4, p. 131])

$$L^2(R) = L^2(R)^\wedge \supset L^2(0,\infty)^\wedge = H^{2+}(R)$$

and the circle (cf. [4, p. 39])

$$L^2(T) = L^2(Z)^\wedge \supset L^2(Z^+)^\wedge = H^{2+}(T).$$

Unfortunately, Beurling's elegant theory of invariant subspaces does not seem to have an analog in the discrete theory, due to the fact that the dual group of Z, $\hat{Z} = T$ is not ordered (cf. [5, Chapter 8, p. 210]).

Define $H^{2-}(z)$ to be the class of discrete analytic function on the lower half-plane $Z^{2-} = \{(m, n); n \leqslant 0\}$ satisfying

$$\sup_{n\leqslant 0} \sum_{m=-\infty}^{\infty} |F(m+in)|^2 < \infty.$$

It is now readily checked that

$$H^{2-}(Z) = L^2(-\pi, 0)^\wedge,$$

so one has the orthogonal decomposition

$$l^2 = L^2(Z) = H^{2+}(Z) \oplus H^{2-}(Z).$$

Let $\chi_{[0,\pi]}$ be the characteristic function of $[0, \pi]$ and let

$$\theta(m, n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[0,\pi]}\, \tilde{e}(it; m + in)\, dt = \frac{1}{2\pi} \int_0^{\pi} \left(\frac{i + ie^{it}}{i + e^{it}}\right)^n e^{imt}\, dt$$
$$= \frac{1}{2\pi} \int_0^{\pi} \tan^n \left(-\frac{t}{2} + \frac{\pi}{4}\right) e^{imt}\, dt$$

which turns out to be Duffin's [2, p. 349] discrete Cauchy kernel. Now, if $F(m, n) \in H^{2+}$ then $F_0^v(t)\, \chi_{[0,\pi]}(t) = F_0^v(t)$ so by (4.4)

$$F_n(m) = \left[F_0^v(t)\, \chi_{[0,\pi]}(t) \left(\frac{1 + ie^{it}}{i + e^{it}}\right)^n\right]^\wedge (m)$$
$$= F_0 * \left[\chi_{[0,\pi]} \left(\frac{1 + ie^{it}}{i + e^{it}}\right)^n\right]^\wedge = F_0 * \theta_n$$

obtaining the following representation formula for H^{2+} functions:

$$F(m + in) = \sum_{m=-\infty}^{\infty} F(k)\, \theta(m - k + in). \tag{4.5}$$

(Compare [2, p. 347, formula 53].)

Another consequence of (4.4) is

$$\sum_{m=-\infty}^{\infty} |F(m+i(n+1))|^2 = \int_0^{\pi} \left| \frac{1+ie^{it}}{i+e^{it}} \right|^{2n+2} |F_0{}^v(t)|^2\, dt$$

$$\leqslant \int_0^{\pi} \left| \frac{1+ie^{it}}{i+e^{it}} \right|^{2n} |F_0{}^v(t)|^2\, dt = \sum_{m=-\infty}^{\infty} |F(m+in)|^2.$$

Thus

$$\|F_n\|^2 = \sum_{m=-\infty}^{\infty} |F(m+in)|^2 \downarrow$$

and

$$\sup_{n\geqslant 0} \left(\sum_{m=-\infty}^{\infty} |F(m+in)|^2 \right)^{1/2} = \left(\sum_{m=-\infty}^{\infty} |F(m)|^2 \right)^{1/2}$$

and we have proved

COROLLARY. *$H^{2+}(Z)$ is a Hilbert space with norm*

$$\|F\| = \sup_{n\geqslant 0} \left(\sum_{m=-\infty}^{\infty} |F(m+in)|^2 \right)^{1/2} = \left(\sum_{m=-\infty}^{\infty} |F(m)|^2 \right)^{1/2}$$

and reproducing kernel $\theta(m+in-k)$.

Finally, let us remark that if we chose to consider $Z_h \times Z_h$ instead of $Z \times Z$ we would have obtained, instead of (4.3), the representation formula

$$F(x+iy) = \int_0^{\pi/h} \tilde{e}_h(it, x+iy) F_0{}^v(t)\, dt$$

which, on letting $h \downarrow 0$ "tends" to the classical Paley–Wiener representation formula:

$$F(x+iy) = \int_0^{\infty} e^{it(\circ+iy)} F_0{}^v(t)\, dt.$$

5. Discrete Paley–Wiener–Schwartz Theorems

Let us recal that a distribution D on the real line R, with compact support, has a Fourier transform $\hat{D}(\xi) = D(e^{ix\xi})$ which can be extended to an entire function $\hat{D}(\zeta) = D(ix\zeta) = D(e^{ix\xi}e^{-x\eta})$, $\zeta = \xi + i\eta$; and one has the following results [3, pp. 210–213]:

(a) Let D be a distribution supported in $[-a, a]$ then the Fourier transform $\hat{D}(\zeta)$ satisfies an inequality of the form

$$|\hat{D}(\zeta)| \leqslant C(1 + |\zeta|)^N e^{a|\eta|}$$

where $\zeta = \xi + i\eta$ and N is the order of D.

(b) The Fourier transform $\hat{\phi}(\zeta)$ of a test function ϕ supported in $[-a, a]$ is an entire function. For each integer k there exists a constant C_k such that

$$|\phi(\zeta)| \leqslant C_k(1 + |\zeta|)^{-k} e^{a|\eta|}.$$

(c) (Converse to (b).) Let $F(\zeta)$ be an entire function with the property that for every integer $k \geqslant 0$ there exists a constant C_k such that

$$|F(\zeta)| \leqslant C_k(1 + |\zeta|)^{-k} e^{|\eta|a}$$

where $\zeta = \xi + i\eta$; then there exists a test function ϕ supported in $[-a, a]$ such that $\hat{\phi}(\zeta) = F(\zeta)$.

(d) (Converse to (a).) Let $F(\zeta)$ be an entire function which satisfies an inequality of the form

$$|F(\zeta)| \leqslant C(1 + |\zeta|)^N e^{a|\eta|};$$

then $F(\zeta)$ is the Fourier transform of a distribution supported in $[-a, a]$.

We were able to prove discrete analogs for (a) and (c) by translating their proofs to the language of the discrete case. However, the proofs of (b) and (d) do not carry over due to the fact that the discrete exponential function is not as nice as the continuous one (in the case of (b)) and to the fact that the multiplication of two discrete analytic functions is not, in general, discrete analytic (in the case of (d)).

Let us consider the exponential $e^{ix\zeta}$ as an entire function of ζ and let x vary along the extended real line $\tilde{R}$; we see that $e^{ix\zeta}$ defines an entire function for each $x \in \tilde{R}\backslash\{\infty, -\infty\}$ $(=R)$ and for each fixed ζ, $e^{ix\zeta}$ behaves nicely as long as one stays away from ∞ and $-\infty$. Now, the discrete exponential function $e(it, m+in)$, $t \in T$ is singular only at $t = \pi/2$ (if $n < 0$) or $t = -(\pi/2)$ (if $n > 0$) so, the pair of points $\{\pi/2, -(\pi/2)\}$ plays the role of the pair $\{\infty, -\infty\}$ in the continuous case. Therefore, compact subsets of $R = \tilde{R}\backslash\{-\infty, \infty\}$ will be replaced by compact subsets of $T\backslash\{-(\pi/2), \pi/2\}$. Indeed, if D is a distribution of T whose support is a compact subset of $T\backslash\{-(\pi/2), \pi/2\}$ then $\hat{D}(m) = D(e^{imt}) = D(\tilde{e}(it; m + i0))$ can be discrete analytically continued to the whole lattice by

$$\hat{D}(m + in) = D(\tilde{e}(it, m + in)). \tag{5.1}$$

This follows from the fact that D is linear and $e(it;\, m + in)$ is discrete analytic for each t in the support of D.

Let us now turn to the statement and proof of the discrete analog of (a).

THEOREM 4. *Let D be distribution on T whose support is a compact subset of $T\backslash\{-(\pi/2), \pi/2\}$ and let it be contained in $\{|\, t \,| \leqslant \alpha\} \cup \{|\, t - \pi \,| \leqslant \alpha\}$, $(0 < \alpha < \pi/2)$; then $\hat{D}(m + in)$ given by* (5.1), *satisfies an inequality of the form*

$$|\, \hat{D}(m + in)| \leqslant K(1 + |\, n \,| + |\, m \,|)^k\, C_\alpha^{|n|} \tag{5.2}$$

where k is the order of D, K is a constant depending only upon D, and

$$C_\alpha = \tilde{e}(-i\alpha;\, 0 + i) = \frac{1 + ie^{-i\alpha}}{i + e^{-i\alpha}}\,.$$

Proof. The proof is similar to the proof of (a) as given in [3, p. 211], only that instead of the nice formula

$$\frac{d}{dt}\, e^{it(m+in)} = (im - n)\, e^{it(m+in)} = ime^{it(m+in)} - ne^{it(m+in)}$$

you have a somewhat more involved equality

$$\frac{d}{dt}\, \tilde{e}(it;\, m + in)$$
$$= im\tilde{e}(it;\, m + in) - \frac{n}{2}\,[\tilde{e}(it;\, m + i(n + 1)) + \tilde{e}(it;\, m + i(n - 1))]$$

from which $(d^k/dt^k)\, \tilde{e}(it;\, m + in)$ can be computed inductively. Beside this minor technical complication the proof is the same.

Let F and G be functions on Z^2 and let Γ: $a = z_0, z_1, \ldots, z_n = b$ denote a discrete contour $(|\, z_{i+1} - z_i \,| = 1, 0 \leqslant i \leqslant m - 1)$. Duffin [2] defined the contour integral

$$\int_\Gamma F\colon G\, \partial z = \sum_{n=1}^{m} (F(Z_n) + F(z_{n-1}))\, (G(z_n) + G(z_{n-1}))\left(\frac{z_n - z_{n-1}}{4}\right) \tag{5.3}$$

and showed that if F and G are discrete analytic in a region containing Γ, and Γ is a closed contour then

$$\int_\Gamma F\colon G\, \partial z = 0. \tag{5.4}$$

Let us turn to the proof of the discrete analog of (c).

THEOREM 5. *Let $F(m+in)$ be a discrete entire function with the property that for every integer $k \geqslant 0$ there is a constant K_k such that*

$$|F(m+in)| \leqslant K_k(1+|n|+|m|)^{-k} C_\alpha^{|n|} \tag{5.5}$$

where

$$C_\alpha = \tilde{e}(-i\alpha; 0+i) = \frac{1+ie^{-i\alpha}}{i+e^{-i\alpha}},$$

then there exists a C^∞ function ϕ supported in $A_\alpha = \{|t| \leqslant \alpha\} \cup \{|\pi - t| \leqslant \alpha\}$ such that

$$F(m+in) = \hat{\phi}(m+in) = \int_{A_\alpha} \phi(t)\, e^{imt} \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n dt. \tag{5.6}$$

Proof. For each n, $F_n(m) = F(m+in)$ decreases faster than any power of $1/|m|$ and thus

$$F_n^v(t) = \sum_{m=-\infty}^{\infty} F_n(m)\, e^{-imt}$$

is a C^∞ function on T for every $n \in Z$, and it is easily checked just as in the proof of Theorem 3 that

$$F_n^v(t) = \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n F_0^v(t) \qquad \forall n \in Z$$

and thus

$$F(m+in) = \int_{-\pi}^{\pi} F_0^v(t)\, e^{imt} \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n dt.$$

It remains to show that $F_0^v(t)$ vanishes outside $A_\alpha = \{|t| \leqslant \alpha\} \cup \{|\pi - t| \leqslant \alpha\}$, i.e., inside, $\{|t-(\pi/2)| < (\pi/2) - \alpha\} \cup \{|t+(\pi/2)| < (\pi/2) - \alpha\}$. Take $\beta \in \{|t-(\pi/2)| < (\pi/2) - \alpha\}$ and consider the discrete contour integral

$$\int_{C_R} \tilde{e}(i\beta; m+in) : F(m+in) \tag{5.7}$$

where the discrete contour is taken to be the boundary of the rectangle $-R \leqslant m \leqslant R$, $0 \leqslant n \leqslant R$. Since both $e(i\beta; m+in)$ and $F(m+in)$ are discrete entire and C_R is a closed contour it follows that the contour integral (5.7) vanishes. But by the definition (5.3)

$$0 = \int_{C_R} \tilde{e}(i\beta; m+in) : F(m+in)$$

$$= \tfrac{1}{4} \sum_{m=-R}^{R} (e^{i\beta m} + e^{i\beta(m+1)})\,(F(m) + F(m+1) + \int_{C_R'} \tilde{e}(i\beta; m+in) : F(m+in)$$

where C_R' is the part of C_R which lies in the "open" halflattice $n > 0$.

By (5.5)

$$|\, e(i\beta;\, m + in)| \cdot |\, F(m + in)| \leqslant K_k(1 + |\, n\,| + |\, m\,|)^{-k}\, (C_\alpha/C_\beta)^n$$

Since $|\,\beta - (\pi/2)| \leqslant (\pi/2) - \alpha$, $C_\beta = e(-i\beta; 0 + i) > C_\alpha$ and $\int_{C_R{}'} \tilde{e}(i\beta;\, m + in)$: $F(m + in)$ tends to zero as $R \to \infty$.

Consequently,

$$\lim_{R\to\infty} \sum_{m=-R}^{R} (e^{i\beta m} + e^{i\beta(m+1)})\,(F(m) + F(m + 1) = 0. \tag{5.8}$$

But

$$\phi(-\beta) = F_0{}^v(-\beta) = \sum_{-\infty}^{\infty} F(m)\, e^{i\beta m}$$

and (5.8) implies that

$$(1 + e^{i\beta})^2 \sum_{m=-\infty}^{\infty} F(m)\, e^{i\beta m} = 0.$$

Since $\beta \neq -(\pi/2)$ it follows that $\phi(-\beta) = 0$ for every β in $|\, t - (\pi/2)| \leqslant (\pi/2) - \alpha$, i.e., ϕ vanishes in $|\, t + (\pi/2)| \leqslant (\pi/2) - \alpha$. If C_R is chosen in the lower half-lattice you get that ϕ vanishes in $|\, t - (\pi/2)| \leqslant (\pi/2) - \alpha$ and thus $F_0{}^v(t) = \phi(t)$ is supported in $A_\alpha = \{|\, t\,| \leqslant \alpha\} \cup \{|\, t - \pi\,| \leqslant \alpha\}$ and (5.6) follows.

References

1. J. Ferrand, Fonctions préharmoniques et fonctions préholomorphes, *Bull. Sci. Math.* **68** (1944), 152–180.
2. R. J. Duffin, Basic properties of discrete analytic functions, *Duke Math. J.* **23** (1956), 335–363.
3. W. F. Donoghue, Jr., "Distributions and Fourier Transforms," Academic Press, New York, 1969.
4. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, N.J., 1962.
5. W. Rudin, "Fourier Analysis on Groups," Interscience, New York, 1962.
6. R. E. Edwards, "Fourier Series," Vol. II, Holt, Rinehart & Winston, New York, 1967.