JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 57, 350–367 (1977)

A New Approach to the Theory of Discrete Analytic Functions

DORON ZEILBERGER

Department of Theoretical Mathematics, The Weizmann Institute of Sciences, Rehovot, Israel

Submitted by R. P. Boas

1. INTRODUCTION

Let $Z \times Z$ be the set of lattice points in the plane

$$\{(m, n); m \text{ and } n \text{ integers}\}.$$

A function $f: Z \times Z \to \mathbb{C}$ is said to be discrete analytic in the lattice square

$$\{(m, n), (m + 1, n), (m + 1, n + 1), (m, n + 1)\}$$

if

$$Lf(m, n) \equiv f(m, n) + if(m + 1, n) + i^2f(m + 1, n + 1) + i^3f(m, n + 1) = 0. \tag{1.1}$$

The concept of a discrete analytic function was introduced by Ferrand [1] and many properties of discrete analytic functions were obtained by Duffin [2]. A convolution product for discrete analytic functions was defined by Duffin and Duris [3] and an operational calculus was developed by Hayabara [4] and further extended by Deeter and Lord [5].

In the present paper a formal power series approach to the theory of discrete analytic functions is given which (it is hoped) not only gives new insight to the theory but also makes many proofs much simpler and shorter. To illustrate the method, new proofs are given to most of the results in [3, 5].

If a function is discrete analytic in a simple region (a finite union of unit squares which is simply connected) it can be discrete analytically continued to the whole plane. Until Section 9 we shall assume that our functions are defined and discrete analytic in each unit square of the quarter plane

$$Z^+ \times Z^+ = \{(m, n); m, n \text{ integers}, m, n \geqslant 0\}.$$

Since functions defined and discrete analytic in the other quarter planes

ISSN 0022-247X

can receive a similar treatment, our assumption involves no loss of generality.

The key idea of this paper is to associate with each function $f: Z^+ \times Z^+ \to \mathbb{C}$ the formal power series

$$\mathbf{f}(X, Y) = \sum_{\substack{m=0\\n=0}}^{\infty} f(m, n) X^m Y^n. \tag{1.2}$$

2. The Ring of Formal Power Series in Two Variables

The class of formal power series

$$R_{XY} = \left\{ \sum_{\substack{m=0\\n=0}}^{\infty} a_{mn} X^m Y^n ; a_{mn} \in \mathbb{C} \right\}$$

endowed with the usual rules for addition and multiplication,

$$\left(\sum a_{mn} X^m Y^n\right) + \left(\sum b_{mn} X^m Y^n\right) = \sum (a_{mn} + b_{mn}) X^m Y^n,$$

$$\left(\sum a_{mn} X^m Y^n\right)\left(\sum b_{mn} X^m Y^n\right) = \sum_{m,n=0}^{\infty} \left(\sum_{k=0}^{m} \sum_{r=0}^{n} a_{k,r} b_{m-k,n-r}\right) X^m Y^m,$$

is a ring with an additive identity 0 ($a_{mn} = 0$ for each m and n) and a multiplicative identity 1 ($a_{00} = 1$, $a_{mn} = 0$ otherwise).

Since the product of any two nonzero formal power series is nonzero, this ring is an integral domain. An element $F(X, Y)$ of R_{XY} has a multiplicative inverse iff $a_{00} = F(0, 0) \neq 0$ and then

$$F(X, Y)^{-1} = \left(a_{00}\left[1 - \frac{[a_{00} - F(X, Y)]}{a_{00}}\right]\right)^{-1} = a_{00}^{-1} \sum_{n=0}^{\infty} \left(\frac{a_{00} - F(X, Y)}{a_{00}}\right)^n;$$

the infinite sum on the right defines a formal power series since the coefficient of each term is a finite sum and there are no problems of convergence. Of course, the inverse, when it exists, is unique, since R_{XY} has no zero divisors.

Later we shall also consider the ring R_X of formal power series of one variable $\{\sum_{n=0}^{\infty} a_n X^n\}$. This is a subring of R_{XY} and also an integral domain.

The following lemma will be needed later:

Lemma 2.1. *Let* $\phi(X) = \sum_{m=0}^{\infty} a_m X^m$. *The equation* $\psi(X)^k = \phi(X)$ *has a* *solution $\psi(X) \in R_X$ iff there exists an integer $n \geq 0$ such that the first nonzero coefficient of $\phi(X)$ is a_{nk}. In this case $\psi(X)$ is given by*

$$\psi(X) = X^n (a_{nk})^{1/k} \{1 + ((a_{nk+1}/a_{nk}) X + (a_{nk+2}/a_{nk}) X^2 + \cdots)\}^{1/k} \quad (2.1)$$

where the right-hand side is developed according to Newton's binomial expansion

$$(1 + \chi)^{1/k} = \sum_{n=0}^{\infty} \binom{1/k}{n} \chi^n.$$

Proof. Verify formally that $\psi(X)^k = \phi(X)$ to prove sufficiency. The necessity is trivial.

3. Representation of Discrete Analytic Functions as Formal Power Series

Let $f: Z^+ \times Z^+ \to \mathbb{C}$ be any function and associate with it the formal power series

$$\mathbf{f}(X, Y) = \sum_{\substack{m=0\\n=0}}^{\infty} f(m, n) X^m Y^n. \quad (1.2)$$

$$(1 + iX - XY - iY)\,\mathbf{f}(X, Y) = (1 + iX)\phi_f(X) + (1 - iY)\psi_f(Y) - f(0,0) - \sum_{\substack{m=0\\n=0}}^{\infty} Lf(m, n) X^{m+1} Y^{n+1}, \quad (3.1)$$

where

$$\phi_f(X) = \sum_{m=0}^{\infty} f(m, 0) X^m \qquad \text{and} \qquad \psi_f(Y) = \sum_{n=0}^{\infty} f(0, n) Y^n.$$

Now the last term vanishes for discrete analytic functions and so for such f

$$(1 + iX - XY - iY)\,\mathbf{f}(X, Y) = (1 + iX)\phi_f + (1 - iY)\psi_f - f(0, 0).$$

Multiplying both sides by $(1 + iX - XY - iY)^{-1}$ yields

$$\mathbf{f}(X, Y) = \frac{\phi_f(X)(1 + iX) + \psi_f(Y)(1 - iY) - f(0,0)}{1 + iX - XY - iY}. \quad (3.2)$$

This confirms the self-evident fact that a discrete analytic function is uniquely determined by its values on the axes. In fact, (3.2) is a condensed form of formula (7) in [2].

Now let

$$\phi_k(X) = \sum_{m=0}^{\infty} f(m, k)\, X^m$$

for $k = 0, 1, 2, \ldots$ so that $\phi_0(X) = \phi_f(X)$ and

$$\mathbf{f}(X, Y) = \sum_{k=0}^{\infty} \phi_k(X)\, Y^k.$$

Introduce this notation into (3.2); comparing coefficients of Y yields

$$\phi_1(X) = \phi_0(X) \frac{X + i}{1 + iX} + \frac{f(0, 1) - if(0, 0)}{1 + iX}, \tag{3.3}$$

and by applying this formula to the function $f^k(m, n) = f(m, n + k)$, we obtain

$$\phi_{k+1}(X) = \phi_k(X) \frac{X + i}{1 + iX} + \frac{f(0, k + 1) - if(0, k)}{1 + iX}. \tag{3.4}$$

Formula (3.4) gives a convenient way to evaluate inductively the values of f inside $Z^+ \times Z^+$ from its values on the axes.

Since a discrete analytic function in $Z^+ \times Z^+$ is uniquely determined by the pair (ϕ_f, ψ_f) and evidently each discrete analytic function determines such a pair, there is a (1–1) correspondence between discrete analytic functions and the elements of the set

$$\{(\phi(X), \psi(Y)); \phi(X) \in R_X, \psi(Y) \in R_Y, \phi(0) = \psi(0)\}.$$

In the following, a discrete analytic function in $Z^+ \times Z^+, f$, will be identified with the pair (ϕ_f, ψ_f) referred to as the "function" (ϕ_f, ψ_f).

EXAMPLE 3.1. The discrete analytic function $f(m, n) \equiv C$ (C constant) corresponds to the pair (ϕ_f, ψ_f), where

$$\phi_f = C \sum_{n=0}^{\infty} X^n = C(1 - X)^{-1},$$

$$\psi_f = C \sum_{n=0}^{\infty} Y^n = C(1 - Y)^{-1},$$

and

$$\mathbf{C}(X, Y) = C \sum_{\substack{m=0\\n=0}}^{\infty} X^m Y^n = \frac{C}{(1-X)(1-Y)}.$$

EXAMPLE 3.2. The function $f(m, n) = C(-1)^{m+n}$ corresponds to the pair

$$\left(C \sum_{n=0}^{\infty} (-1)^n X^n, C \sum_{n=0}^{\infty} (-1)^n Y^n\right) = \left(\frac{C}{1+X}, \frac{C}{1+Y}\right)$$

and

$$\mathbf{f}(X, Y) = C \sum_{\substack{m=0\\n=0}}^{\infty} (-1)^{m+n} X^m Y^n = \frac{C}{(1+X)(1+Y)}.$$

Duffin [2] termed a function f which assumes the value e_1 on the odd lattice points and the value e_2 on the even lattice points, a biconstant. This function can be written as

$$\tfrac{1}{2}(e_2 + e_1) + (-1)^{m+n}\, \tfrac{1}{2}(e_2 - e_1).$$

Thus, the general form of a biconstant is

$$\frac{1}{2}(e_2 + e_1)\left(\frac{1}{1-X}, \frac{1}{1-Y}\right) + \frac{1}{2}(e_2 - e_1)\left(\frac{1}{1+X}, \frac{1}{1+Y}\right)$$

$$= \left(\frac{e_2 + e_1 X}{(1-X)(1+X)}, \frac{e_2 + e_1 Y}{(1-Y)(1+Y)}\right).$$

4. INTEGRAL AND DERIVATIVE

Duffin [2] defined a "line integral" by the rule

$$\int_a^b f(z)\, \partial z = \sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1})/2, \tag{4.1}$$

where $a = z_0, z_1, \ldots, z_m = b$ is a chain of lattice points (that is, $|z_k - z_{k+1}| = 1$ and $f_k = f(z_k)$).

He showed that if f is discrete analytic in a region then the sum is independent of the particular chain connecting a to b and hence (4.1) is well defined. He defined the indefinite integral F of f,

$$F(z) = \int_a^z f(z)\, \partial z. \tag{4.2}$$

Since the starting point of the integral is arbitrary, $F(z)$ is only defined up to an additive constant. Duffin also showed that if $f(z)$ is discrete analytic in a simple region then so is $F(z)$.

Now, suppose $f = (\phi_f, \psi_f)$ and $F = (\phi_F, \psi_F)$. We would then like to find ϕ_F in terms of ϕ_f and ψ_F in terms of ψ_f.

By (4.1) with $a = 0$ we have

$$\begin{aligned} 2F(m, 0) &= f(0, 0) + 2f(1, 0) + \cdots + 2f(m-1, 0) + f(m, 0) \\ &= 2[f(0, 0) + \cdots + f(m, 0)] - (f(0, 0) + f(m, 0)). \end{aligned}$$

Thus

$$2 \sum F(m, 0) X^m = 2(1/(1-X))\phi_f(X) - \phi_f(X) - (f(0,0)/(1-X))$$

and we get

$$\phi_F = \frac{1}{2} \frac{1+X}{1-X} \phi_f - \frac{f(0,0)}{2(1-X)}. \tag{4.3a}$$

Similarly,

$$\psi_F = \frac{i}{2} \frac{1+Y}{1-Y} \psi_f - \frac{if(0,0)}{2(1-Y)}. \tag{4.3b}$$

Thus, the operation of integration is

$$\begin{aligned} (\phi(X), \psi(Y)) \to \frac{1}{2} \left(\frac{1+X}{1-X} \phi(X) - \frac{f(0,0)}{1-X}, i \frac{1+Y}{1-Y} \psi(Y) - \frac{if(0,0)}{1-Y} \right) \\ + C \left(\frac{1}{1-X}, \frac{1}{1-Y} \right), \end{aligned} \tag{4.4}$$

where C is an arbitrary constant. If the starting point of integration in (4.2) $a = 0$, then $F(z) = \int_0^z f(z)\, \partial z$, and, in (4.4), $C = 0$.

Duffin also defined the dual f^- of discrete functions by the rule $f^-(m, n) = (-1)^{m+n} f^*(m, n)$. Thus $\mathbf{f}^-(X, Y) = \mathbf{f}^*(-X, -Y)$ and $(\phi(X), \psi(Y))^- = (\phi^*(-X), \psi^*(-Y))$. We are now in a position to give another proof of the following result, which was first proved in [2, p. 341].

LEMMA 4.1. *Let $F(z)$ be a given discrete analytic function. Let a and b be points of $Z^+ \times Z^+$ and let k be an arbitrary constant. Then*

$$f(z) = \left(4 \int_b^z F^- \partial z + k\right)^- \tag{4.5}$$

is analytic in $Z^+ \times Z^+$ and

$$F(z) = \int_a^z f(z)\, \partial z + F(a). \tag{4.6}$$

Proof. If $F = (\Phi, \Psi)$, then $F^- = (\Phi^*(-X), \Psi^*(-Y))$.
By (4.4),

$$4 \int_b^z F^- \partial z + k$$

$$= 2 \left(\Phi^*(-X) \frac{1+X}{1-X} - \frac{\Phi^*(0)}{1-X}, i\Psi^*(-Y) \frac{1+Y}{1-Y} - \frac{i\Psi^*(0)}{1-Y} \right)$$

$$+ k_1 \left(\frac{1}{1-X}, \frac{1}{1-Y} \right)$$

(k_1 some other constant).
So,

$$f(z) = \left(4 \int_b^z F^- \partial z + k \right)^-$$

$$= 2 \left(\Phi(X) \frac{1-X}{1+X} - \frac{\Phi(0)}{1+X}, -i\Psi(Y) \frac{1-Y}{1+Y} + \frac{i\Psi(0)}{1+Y} \right) \qquad (4.7)$$

$$+ k_1^* \left(\frac{1}{1+X}, \frac{1}{1+Y} \right).$$

Finally,

$$\int_0^z f(z)\, \partial z = \left[\left(\Phi(X) \frac{1-X}{1+X} - \frac{\Phi(0)}{1+X} \right) \frac{1+X}{1-X}, \right.$$

$$\left. - i \left(\Psi(Y) \frac{1-Y}{1+Y} - \frac{\Psi(0)}{1+Y} \right) \frac{i(1+Y)}{(1-Y)} \right]$$

$$+ k^* \underbrace{\left(\frac{1}{1+X} \cdot \frac{1+X}{1-X} - \frac{1}{1-X}, \frac{i}{1+Y} \frac{1+Y}{1-Y} - \frac{1}{1-Y} \right)}_{0}$$

$$= \left(\Phi(X) - \frac{F(0,0)}{1-X}, \Psi(Y) - \frac{F(0,0)}{1-Y} \right).$$

Thus,

$$F(z) = \int_0^z f(z)\, \partial z + F(0,0).$$

Duffin [2] used formula (4.5) to define $f = \partial F/\partial z$ as the derivative of F. Formula (4.7) says that the action of taking the derivative is

$$(\phi(X), \psi(Y)) \to 2\left(\phi(X)\frac{1-X}{1+X} - \frac{\phi(0)}{1+X}, -i\left(\psi(Y)\frac{1-Y}{1+Y} - \frac{\psi(0)}{1+Y}\right)\right) + k\left(\frac{1}{1+X}, \frac{1}{1+Y}\right) \tag{4.7}$$

(k an arbitrary constant).
So the derivative is unique up to addition by a constant multiple of $(-1)^{m+n}$ (Example 3.2).

5. Polynomials[1]

In [2, Sect. 5], polynomials which are discrete analytic everywhere were considered and it was shown if f is a discrete analytic polynomial then the integral F is a discrete analytic polynomial. A sequence of discrete analytic polynomials was defined by the relations

$$z^{(n+1)} = (n+1)\int_0^z z^{(n)}\,\partial z; \qquad z^{(0)} \equiv 1. \tag{5.1}$$

So,

$$z^{(0)} = \left(\frac{1}{1-X}, \frac{1}{1-Y}\right)$$

$$z^{(1)} = \frac{1}{2}\left(\frac{1+X}{1-X}\cdot\frac{1}{1-X} - \frac{1}{1-X}, \frac{i(1+Y)}{(1-Y)}\cdot\frac{1}{1-Y} - \frac{i}{1-Y}\right)$$

$$= \left(\frac{X}{(1-X)^2}, \frac{iY}{(1-Y)^2}\right).$$

Since $z^{(k)}(0) = 0$, $k = 1, 2, 3, \ldots$, one gets

$$z^{(n)} = \left(\frac{n!}{2^{n-1}}\frac{X(1+X)^{n-1}}{(1-X)^{n+1}}, \frac{(i)^n\, n!}{2^{n-1}}\frac{Y(1+Y)^{n-1}}{(1-Y)^{n+1}}\right), \qquad n = 1, 2, 3, \ldots \tag{5.2}$$

The discrete analytic exponential function

$$e(z, t) = ((2+t)/(2-t))^x\,((2+it)/(2-it))^y$$

[1] By $f(z)$ we mean $f(x, y)$, where $z = x + iy$, $(x, y) \in Z \times Z$.

was introduced by Ferrand [1] and it is seen that

$$e(z,t)=\left(\frac{1}{1-((2+t)/(2-t))\,X},\ \frac{1}{1-((2+it)/(2-it))\,Y}\right)$$

and

$$\mathbf{e}(z,t)=\frac{\left(\dfrac{1+iX}{1-((2+t)/(2-t))\,X}+\dfrac{1-iY}{1-((2+it)/(2-it))}-1\right)}{1}$$

$$=\frac{1}{(1-((2+t)/(2-t))\,X)\,(1-((2+it)/(2-it))\,Y)}.$$

By using (5.2) one can reprove [2, formula 139],

$$e(z,t)=\sum_{n=0}^{\infty}(z^{(n)}t^n/n!)\qquad (|\,t\,|<2).$$

6. A Convolution Product for Discrete Function Theory

In [3], three types of convolution products were defined for discrete analytic functions.

The convolution of f, g is defined as

$$f*g=\int_0^z f(z-t):g(t)\,\partial t, \tag{6.1}$$

where

$$\int_a^b f(z):g(z)\,\partial z=\sum_{n=1}^{m}\tfrac{1}{4}[f(z_n)+f(z_{n-1})]\cdot[g(z_n)+g(z_{n-1})]\cdot(z_n-z_{n-1}),$$

where $a=z_0, z_1, \ldots, z_n=b$ is a chain connecting a and b.

It was shown in [3] that if f, g are discrete analytic then so is $f*g$.

For $\phi(X)\in R_X$ define

$$\overline{\phi(X)}=((1+X)\phi(X)-\phi(0))/X.$$

Then, in terms of $f=(\phi_f, \psi_f), g=(\phi_g, \psi_g)$; $f*g=(\phi_{f*g}, \psi_{f*g})$ is given by

$$\phi_{f*g}=\tfrac{1}{4}X\bar{\phi}_f\bar{\phi}_g,\qquad \psi_{f*g}=\tfrac{1}{4}iY\bar{\psi}_f\bar{\psi}_g. \tag{6.2}$$

Also, $\Phi_{f*g}(0)=0$, so $\bar{\phi}_{f*g}=\frac{1}{4}(1+X)\bar{\phi}_f\bar{\phi}_g$ and if h is discrete analytic,

$$\phi_{(f*g)*h}=(X/4)\,\bar{\phi}_{f*g}\bar{\phi}_h=(X(1+X)/16)\,\bar{\phi}_f\bar{\phi}_g\bar{\phi}_h=\phi_{f*(g*h)}.$$

Similarly, $\psi_{(f*g)*h}=\psi_{f*(g*h)}$ and we obtain a simple proof of the result of [3] that the convolution product is associative: $(f*g)*h=f*(g*h)$.

Invoking (5.2),

$$\begin{aligned}\phi_{(z^{(n)}/n!)*(z^{(m)}/m!)} &= \frac{1}{4}\,\frac{1}{2^{n-1}}\cdot\frac{1}{2^{m-1}}\,\frac{X(1+X)^{n-1}(1+X)\,X(1+X)^{m-1}(1+X)}{(1-X)^{n+1}(1-X)^{m+1}\,X} \qquad (n, m \geqslant 1),\\ &= \frac{1}{2^{n+m}}\,\frac{X(1+X)^{n+m}}{(1-X)^{n+m+2}} = \phi_{z^{(n+m+1)}/(n+m+1)!}\,.\end{aligned}$$

Similarly,

$$\psi_{(z^{(n)}/n!)*(z^{(m)}/m!)} = \psi_{z^{(n+m+1)}/(n+m+1)!}\,.$$

Thus [3, p. 205];

$$(z^{(n)}/n!) * (z^{(m)}/m!) = z^{(n+m+1)}/(n+m+1)!.$$

The Prime Convolution Product

The prime convolution product of $f(z)$ and $g(z)$ was defined in [3] to be

$$f *' g = \int_0^z f(z-t) : g'(t)\,\partial t + f(z)\,g(0), \tag{6.3}$$

where

$$\int_a^b f : g'\,\partial z = \tfrac{1}{2}\sum_{n=1}^{\infty} (f(z_n) + f(z_{n-1}))\,(g(z_n) - g(z_{n-1}))$$

and it was shown there that if f, g is discrete analytic, so is $f *' g$.

The coefficient of X^{n-1} in $\phi_{f*'g}$ is

$$\begin{aligned}&\tfrac{1}{2}[f(n) + f(n-1)]\,[g(1) - g(0)] + \tfrac{1}{2}[f(n-1) + f(n-2)]\cdot[g(2) - g(1)]\\ &\quad + \cdots + [f(1) + f(0)]\,[g(n) - g(n-1)] + f(n-1)\,g(0).\end{aligned}$$

Thus,

$$\begin{aligned}\phi_{f*'g} &= \frac{1}{2}\,\frac{(1+X)\,\phi_f - \phi_f(0)}{X}\cdot\frac{(1-X)\,\phi_g - \phi_g(0)}{X}\cdot X + \phi_g(0)\,\phi_f\\ &= \tfrac{1}{2}X\bar{\phi}_f(\bar{\phi}_g - 2\phi_g) + \phi_g(0)\,\phi_f = \tfrac{1}{2}X\bar{\phi}_f\bar{\phi}_g - X\phi_g\bar{\phi}_f + \phi_g(0)\,\phi_f\\ &= \frac{1}{2}\,X\bar{\phi}\,\bar{\phi}_g - X\phi_g\bar{\phi}_f + \phi_g(0)\,\frac{X\bar{\phi}_f + \phi_f(0)}{1+X}\\ &= \frac{1}{2}\,X\bar{\phi}_f\bar{\phi}_g - X\bar{\phi}_f\left[\phi_g - \frac{\phi_g(0)}{1+X}\right] + \frac{\phi_g(0)\,\phi_f(0)}{1+X}\\ &= \frac{1}{2}\,X\bar{\phi}_f\bar{\phi}_g - X\bar{\phi}_f\,\frac{X\bar{\phi}_g}{1+X} + \frac{\phi_f(0)\,\phi_g(0)}{1+X}\\ &= \frac{X(1-X)}{2(1+X)}\bar{\phi}_f\bar{\phi}_g + \frac{\phi_f(0)\,\phi_g(0)}{1+X}\,.\end{aligned}$$

Similarly,

$$\psi_{f*'g} = \frac{Y(1-Y)}{2(1+Y)} \bar{\psi}_f \bar{\psi}_g + \frac{\psi_f(0)\,\psi_g(0)}{1+Y}.$$

Thus,

$$\overline{\phi_{f*'g}} = ((1-X)/2)\,\bar{\phi}_f\bar{\phi}_g \qquad \text{and} \qquad \overline{\psi_{f*'g}} = ((1-Y)/2)\,\bar{\psi}_f\bar{\psi}_g. \tag{6.4}$$

Now, $\phi(X),\ \psi(X) \in R_X,\ \overline{\phi(X)} = \overline{\psi(X)} \Rightarrow \phi(X) - \psi(X) = \text{constant}$. Since $f *' g(0) = f(0)\,g(0) = g *' f(0)$ and

$$\overline{\phi_{f*'g}} = ((1-X)/2)\,\bar{\phi}_f\bar{\phi}_g = \overline{\phi_{g*'f}}$$

(and similarly $\overline{\psi_{f*'g}} = \overline{\psi_{g*'f}}$), it is seen that the prime convolution product is commutative.

Also, from (6.4),

$$\overline{\phi_{(f*'g)*'h}} = ((1-X)/2)^2\,\bar{\phi}_f\bar{\phi}_g\bar{\phi}_h = \overline{\phi_{f*'(g*'h)}},$$

and $[(f *' g) *' h]\,(0) = f(0)\,g(0)\,h(0) = [f *' (g *' h)]\,(0)$, it follows that $(f *' g) *' h = f *' (g *' h)$ and the associativity of the prime convolution product is proved.

Let us prove that

$$\frac{z^{(n)}}{n!} *' \frac{z^{(m)}}{m!} = \frac{z^{(n+m)}}{(n+m)!} \qquad (n, m \geqslant 1),$$
$$\overline{\phi_{z^{(n)}/n!}} = \frac{1}{2^{n-1}} \frac{(1+X)^n}{(1-X)^{n+1}} \qquad (n \geqslant 1). \tag{6.5}$$

So,

$$\overline{\phi_{(z^{(n)}/n!)*'\,z^{(m)}/m!)}} = \frac{1}{2^{n-1}} \frac{(1+X)^n}{(1-X)^{n+1}} \cdot \frac{1}{2^{m-1}} \frac{(1+X)^m}{(1-X)^{m+1}} \cdot \frac{(1-X)}{2}$$
$$= \frac{1}{2^{n+m-1}} \cdot \frac{(1+X)^{n+m}}{(1-X)^{n+m+1}} = \overline{\phi_{z^{(n+m)}/(n+m)!}}.$$

Similarly,

$$\overline{\psi_{(z^{(n)}/n!)*'(z^{(m)}/m!)}} = \overline{\psi_{z^{(n+m)}/(n+m)!}}.$$

Of course,

$$\frac{z^{(n)}}{n!} *' \frac{z^{(m)}}{m!}(0) = \frac{z^{(n+m)}}{(n+m)!}(0),$$

giving (6.5).

The Double Prime Convolution Product

The double prime product was defined in [3] by

$$f *'' g = \int_0^z (\partial f(z-t)/\partial z) : (\partial g(t)/\partial t)\, \partial t,$$

where $\partial f/\partial z$ is the discrete analytic derivative of f, defined in Section 4. Invoking formulas (4.7) and (6.2) we get

$$\phi_{f*''g} = \left(\frac{(1-X)\phi_f - \phi_f(0)}{4X}\right) \cdot \left(\frac{(1-X)\phi_g - \phi_g(0)}{X}\right) \cdot X.$$

For $\phi \in R_X$, let

$$\tilde{\phi} = ((1-X)\phi - \phi(0))/X.$$

Then,

$$\phi_{f*''g} = (X/4)\tilde{\phi}_f\tilde{\phi}_g, \qquad \psi_{f*''g} = (iY/4)\tilde{\psi}_f\tilde{\psi}_g, \tag{6.6}$$

from which the commutativity of the double prime product is seen. Also,

$$\phi_{(f*''g)*''h} = \frac{X}{4}\tilde{\phi}_f\tilde{\phi}_g \frac{(1-X)}{X} \frac{\tilde{\phi}_h X}{4} = \frac{X(1-X)}{16}\tilde{\phi}_f\tilde{\phi}_g\tilde{\phi}_h = \phi_{f*''(g*''h)}.$$

Similarly for the ψ's; therefore,

$$(f *'' g) *'' h = f *'' (g *'' h).$$

7. New Proofs to Some Results of Deeter and Lord [5]

In this section it will be shown that Deeter and Lord's Theorem 1, Lemma 2, and Theorem 7 [5] (here, Propositions 7.1, 7.2, 7.3, respectively) are immediate consequences of formula (6.4).

Deeter and Lord [5] defined the mean of the function on the positive x-axis and y-axis, respectively, by

$$\bar{f}(m, 0) = \tfrac{1}{2}[f(m, 0) + f(m-1, 0)], \qquad m = 1, 2, \ldots,$$
$$\bar{f}(0, m) = \tfrac{1}{2}[f(0, m) + f(0, m-1)], \qquad m = 1, 2, \ldots.$$

So, in our notation,

$$\sum_{m=1}^{\infty} \bar{f}(m, 0)\, X^m = X\tilde{\phi}_f, \qquad \sum_{m=1}^{\infty} \bar{f}(0, m)\, Y^m = Y\tilde{\psi}_f. \tag{7.1}$$

If f has mean zero on the $x(y)$ axis then $\tilde{\phi}_f(\tilde{\psi}_f) \equiv 0$.

PROPOSITION 7.1 (Theorem 1 in [5]). *If two discrete analytic functions are such that the mean of either function is zero on an axis then the mean of their (prime convolution) product is zero on that axis.*

Proof. Immediate from formula (6.4).

PROPOSITION 7.2 (Lemma 2 in [5]). *Let f, g be discrete analytic and satisfy*

$$\bar{f}(1,0) = \cdots = \bar{f}(n-1,0) = 0, \qquad \bar{f}(n,0) \neq 0,$$
$$\bar{g}(1,0) = \cdots = \bar{g}(m-1,0) = 0, \qquad \bar{g}(m,0) \neq 0;$$

then

$$\overline{f *' g}(1,0) = \cdots = \overline{f *' g}(m+n-2) = 0$$

and

$$\overline{f *' g}(m+n-1,0) = \bar{f}(n,0)\,\bar{g}(m,0) \neq 0.$$

Proof. From (7.1), the leading term of $\bar{\phi}_f$ is $\bar{f}(n,0)\,X^{n-1}$, the leading term of $\bar{\phi}_g$ is $\bar{g}(m,0)\,X^{m-1}$, and thus the leading term of

$$X\overline{\phi_{f*'g}} = X((1-X)/2)\,\bar{\phi}_f\bar{\phi}_g$$

is $\bar{f}(n,0)\,\bar{g}(m,0)\,X^{n+m-1}$ and the conclusion follows from (7.1).

PROPOSITION 7.3 (Theorem 7 in [5]). *Let f be discrete analytic. A necessary and sufficient condition for the existence of a solution of the equation $g^{*'k} = f$ is that there exist nonnegative integers m and n such that*

$$\bar{f}(1,0) = \cdots = \bar{f}(km,0) = 0, \qquad \bar{f}(km+1,0) \neq 0,$$
$$\bar{f}(0,1) = \cdots = \bar{f}(0,kn) = 0, \qquad \bar{f}(0,kn+1) \neq 0.$$

Proof. Since

$$\overline{\phi_{g*'k}} = ((1-X)/2)^{k-1}\,(\bar{\phi}_g)^k \qquad \text{and} \qquad \overline{\psi_{g*'k}} = ((1-Y)/2)^{k-1}\,(\bar{\psi}_g)^k,$$

then conclusion follows from Lemma 2.1.

8. DISCRETE VOLTERRA INTEGRAL EQUATIONS

Let $f(z)$ and $k(z)$ be discrete analytic functions in a rectangular region R which (without loss of generality) will be assumed to be $Z^+ \times Z^+$. Duffin and

Duris [3] considered the problem of finding a discrete analytic solution $u(z)$ to the equation

$$u(z) = f(z) + \lambda \int_0^z k(z-t) \colon u(t)\, \partial t, \tag{8.1}$$

where λ is an arbitrary constant. Translated to our language, (8.1) reads,

$$\phi_u = \phi_f + (\lambda/4)\, X\bar{\phi}_k\bar{\phi}_u\,, \tag{8.2a}$$

$$\psi_u = \psi_f + (\lambda/4)\, Yi\bar{\psi}_k\bar{\psi}_u\,. \tag{8.2b}$$

Thus

$$\phi_u = \phi_f + (\lambda\bar{\phi}_k/4)\, [\phi_u \cdot (1 + X) - \phi_u(0)], \tag{8.3a}$$

$$\psi_u = \psi_f + (\lambda Yi/4)\, \bar{\psi}_k[\psi_u \cdot (1 + Y) - \psi_u(0)]. \tag{8.3b}$$

Now, if a solution $u(z)$ to (8.1) exists, $\phi_u(0) = \psi_u(0) = f(0)$ by (8.1). So,

$$\phi_u[1 - (\lambda/4)\, \bar{\phi}_k \cdot (1 + X)] = \phi_f - (\lambda\bar{\phi}_k/4)\, f(0), \tag{8.4a}$$

$$\psi_u[1 - (i\lambda/4)\, \bar{\psi}_k \cdot (1 + Y)] = \psi_f - (i\lambda\bar{\psi}_k/4)\, f(0). \tag{8.4b}$$

THEOREM 8.1 (Theorem 5.2 in [3]). *Let $f(z)$ and $k(z)$ be discrete analytic functions in $Z^+ \times Z^+$. Then there exists a unique function $u(z)$ discrete analytic in $Z^+ \times Z^+$ such that*

$$u(z) = f(z) + \lambda \int_0^z k(z-t) \colon u(t)\, \partial t \tag{8.1}$$

for all values except possibly $\lambda = 4/h[k(0) + k(h)]$ where h equals $+1$, $+i$.
(This is not exactly the original wording but it is equivalent to it.)

Proof. A solution of (8.1) exists iff there is a solution of (8.4a) and (8.4b) simultaneously. A solution of (8.4) exists (and then is unique) if the coefficient terms of both

$$1 - (\lambda/4)\, \bar{\phi}_k(1 + X) \qquad \text{and} \qquad 1 - (i\lambda/4)\, \bar{\psi}_k(1 + Y)$$

are not zero; i.e., if

$$1 \neq (\lambda/4)\, (k(0) + k(1)) \qquad \text{and} \qquad 1 \neq (i\lambda/4)\, (k(0) + k(i)),$$

and in this case the unique solution $u = (\phi_u\,, \psi_u)$ is given by

$$\phi_u = (\phi_f - (\lambda\bar{\phi}_k/4)\, f(0))\, [1 - (\lambda/4)\, \bar{\phi}_k(1 + X)]^{-1},$$
$$\psi_u = (\psi_f - (i\lambda\bar{\psi}_k/4)\, f(0))\, [1 - (i\lambda/4)\, \bar{\psi}_k(1 + Y)]^{-1}.$$

If the condition on λ is not satisfied, i.e.,

$$\lambda = 4/[h \cdot (k(0) + k(h))],$$

for either $h = 1$ or $h = i$. Then a solution may or may not exist according to the leading term of the r.h.s. of (8.4). The solution is not unique iff

$$1 - (\lambda/4)\bar{\phi}_k(1 + X) \equiv 0 \qquad \text{or} \qquad 1 - (i\lambda/4)\bar{\psi}_k \cdot (1 + Y) \equiv 0;$$

i.e.,

$$\phi_k = (4/\lambda)(X/(1 + X)^2) + (\phi_k(0)/(1 + X))$$

or

$$\psi_k = (4/i\lambda)(Y/(1 + Y)^2) + (\psi_k(0)/(1 + Y)).$$

It is also possible to prove, by the method of this paper, most of the results in [6].

9. A Representation Formula for the Half-Plane

Consider the abelian group T_{xy} of all formal power series

$$\sum_{m,n=-\infty}^{\infty} a_{mn} X^m Y^n \tag{9.1}$$

(note that we allow here also negative powers). Define, again,

$$\Sigma a_{mn} X^m Y^n + \Sigma b_{mn} X^m Y^n = \Sigma(a_{mn} + b_{mn}) X^m Y^n.$$

Let

$$A = \Sigma a_{mn} X^m Y^n; \qquad B = \Sigma b_{mn} X^m Y^n.$$

$C = AB$ is said to exist if

$$C_{mn} = \sum_{\substack{k=-\infty \\ r=-\infty}}^{\infty} b_{r,k} a_{m-r,n-k}$$

converges absolutely for each m, n integers.

The following lemma is trivial

Lemma 9.1. *If A, B, $C \in T_{XY}$ and B has only a finite number of nonzero terms and if both $(AB)\,C$ and $A(BC)$ exist, then $(AB)\,C = A(BC)$.*

Now we can re-porve the following representation formula from [2, p. 347].

THEOREM 9.2. *Let $q(z)$ be a discrete function such that*

$$Lq(z) = 0, \qquad z \neq 0, \qquad Lq(0) = 1 \tag{9.2}$$

and let $f(z)$ be a function which is analytic in every unit square of the upper half-plane; suppose that for each fixed z_0

$$f(z)\, q(z - z_0) = o\,|\, z\,|^{-1}, \qquad \operatorname{Im} z \geqslant 0. \tag{9.3}$$

Then if

$$\theta(z) = q(-z) + iq(1 - z), \tag{9.4}$$

we have

$$f(z_0) = \sum_{m=-\infty}^{\infty} f(m)\, \theta(z_0 - m), \qquad \operatorname{Im} z_0 \geqslant 0,$$

and the r.h.s. is zero for $\operatorname{Im} z_0 < 0$.

Proof. Let $\mathbf{q} = \sum_{m,n=-\infty}^{\infty} q(m, n)\, X^m Y^n$; from (9.2), $(1 + iX^{-1} - X^{-1}Y^{-1} - iY^{-1})\, \mathbf{q} = 1$. Let $\tilde{q}(z) = q(-z)$; then $\tilde{\mathbf{q}} = \mathbf{q}(X^{-1}, Y^{-1})$. Thus, $(1 + iX - XY - iY)\, \tilde{\mathbf{q}} = 1$. Now

$$\theta(z) = q(-z) + iq(1 - z) = \tilde{q}(z) + i\tilde{q}(-1 + z), \qquad \boldsymbol{\theta} = (1 + iX)\, \tilde{\mathbf{q}}.$$

So,

$$(1 + iX - XY - iX)\, \boldsymbol{\theta} = 1 + iX. \tag{9.6}$$

Let

$$\mathbf{f}(X, Y) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} f(in + m)\, X^m Y^n,$$

$$\phi_0(X) = \sum_{n=-\infty}^{\infty} f(n)\, X^n.$$

Since f is discrete analytic in the upper half-plane, considerations similar to those in Section 3 show that

$$\mathbf{f}(X, Y)\,(1 + iX - XY - iY) = \phi_0(X)\,(1 + iX). \tag{9.7}$$

Multiply both sides by $\boldsymbol{\theta}(X, Y)$. (The product exists by virtue of condition (9.3).)

$$[\mathbf{f}(X, Y)(1 + iX - XY - iX)]\,\boldsymbol{\theta}(X, Y) = [\phi_0(X)(1 + iX)]\,\boldsymbol{\theta}(X, Y).$$

By Lemma 9.1,

$$\mathbf{f}(X, Y)\,[(1 + iX - XY - iY)\,\boldsymbol{\theta}(X, Y)] = \phi_0(X)\,[(1 + iX)\,\boldsymbol{\theta}(X, Y)],$$

since all products involved exist. By (9.6),

$$\begin{aligned}\mathbf{f}(X, Y)(1 + iX) &= \phi_0(X)\,[(1 + iX)\,\boldsymbol{\theta}(X, Y)] = \phi_0(X)\,[\boldsymbol{\theta}(X, Y)(1 + iX)]\\ &= [\phi_0(X)\,\boldsymbol{\theta}(X, Y)]\,(1 + iX).\end{aligned}$$

Let

$$\mathbf{F}(X, Y) = \mathbf{f}(X, Y) - \phi_0(X)\,\boldsymbol{\theta}(X, Y). \tag{9.8}$$

We want to show $F(X, Y) \equiv 0$ and then it would follow that

$$\mathbf{f}(X, Y) = \phi_0(X)\,\boldsymbol{\theta}(X, Y), \tag{9.9}$$

which is the same as (9.5). From (9.8),

$$\mathbf{F}(X, Y)(1 + iX) \equiv 0. \tag{9.10}$$

But

$$\begin{aligned}&\mathbf{F}(X, Y)(1 + iX - XY - iY)\\ &\quad= \mathbf{f}(X, Y)(1 + iX - XY - iY) - [\phi_0(X)\,\boldsymbol{\theta}(X, Y)]\,(1 + iX - XY - iY)\\ &\quad= \phi_0(X)(1 + iX) - \phi_0(X)\,[\boldsymbol{\theta}(X, Y)(1 + iX - XY - iY)]\\ &\quad= \phi_0(X)(1 + iX) - \phi_0(X)(1 + iX) = 0.\end{aligned}$$

So

$$\mathbf{F}(X, Y)(1 + iX - XY - iY) \equiv 0. \tag{9.11}$$

Multiply Eq. (9.10) by $(1 + iY)$ and subtract from (9.11) to get $\mathbf{F}(X, Y)\,2iY \equiv 0$ and consequently, $\mathbf{F}(X, Y) \equiv 0$.

Acknowledgment

The author would like to thank Dr. Harry Dym for his help and encouragement.

REFERENCES

1. J. FERRAND, Fonctions préharmoniques et fonctions préholomorphes, *Bull. Sci. Math.* **68** (1944), 152–180.
2. R. J. DUFFIN, Basic properties of discrete analytic functions, *Duke Math. J.* **23** (1956), 335–363.
3. R. J. DUFFIN AND C. S. DURIS, A convolution product for discrete analytic function theory, *Duke Math. J.* **31** (1964), 199–220.
4. S. HAYABARA, Operational calculus on the discrete analytic functions, *Math. Japon.* 11 (1966), 35–65.
5. C. R. DEETER AND M. E. LORD, Further theory of operational calculus on discrete analytic functions, *J. Math. Anal. Appl.* **26** (1969), 92–113.
6. R. J. DUFFIN AND C. S. DURIS, Discrete analytic continuation of solutions of difference equations, *J. Math. Anal. Appl.* **9** (1964), 252–267.