A VARIATION ON A VERY FAMILIAR ALGORITHM

Author(s): DORON ZEILBERGER and PHILIP TODD
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A VARIATION ON A VERY FAMILIAR ALGORITHM

By DORON ZEILBERGER
University of Illinois
Urbana, IL 61801

and PHILIP TODD
Georgia Institute of Technology
Atlanta, GA 30332

If you asked your students to do the long multiplication 739 × 999, what technique would they employ? A student entrenched in the rigid tradition of long multiplication would probably do it this way:

$$\begin{array}{r} 739 \\ \times\, \underline{999} \\ 6651 \\ 6651 \\ \underline{6651} \\ 738261 \end{array}$$

A student with a little more insight might employ the following technique:

$$739 \times 999 = 739 \times (1000 - 1) = \begin{array}{r} 739000 \\ \underline{-\,739} \\ 738261 \end{array}$$

The second approach seems better than the first because it is simpler to multiply by 1 than by 9. This technique is a very special case but does beg the question, Is it possible to generalize this second technique and produce an algorithm for all multiplication problems?

The usual way of representing a number like 17 is

$$17 = 1 \cdot 10^1 + 7 \cdot 10^0.$$

An alternate way of thinking of 17 is to record it as (1, 7). In this light the standard multiplication table can be redrafted (table 1).

One-digit numbers may also be thought of as ordered pairs, for example,

$$6 = 1 \cdot 10^1 - 4 \cdot 10^0,$$

so

$$6 \rightarrow (1, -4);$$

and

$$7 = 1 \cdot 10^1 - 3 \cdot 10^0,$$

so

$$7 \rightarrow (1, -3).$$

Similarly,

$$\begin{aligned} 8 &\rightarrow (1, -2), \\ 9 &\rightarrow (1, -1), \\ 14 &\rightarrow (1, 4), \\ 15 &\rightarrow (1, 5), \end{aligned}$$

and

$$16 \rightarrow (2, -4).$$

The major benefit of the following algorithm for long multiplication is that it only relies on the following short multiplication table together with the rule $(-m)n = -(mn)$ (see table 2).

Examples

1. $97 \cdot 86$

$$97 = 100 - 3 = 1 \cdot 10^2 + 0 \cdot 10^1 - 3 \cdot 10^0 = (1, 0, -3);$$

$$86 = 100 - 14 = 1 \cdot 10^2 - 1 \cdot 10^1 - 4 \cdot 10^0 = (1, -1, -4).$$

Thus, the product $97 \cdot 86$ can be calculated as follows:

$$\begin{array}{rrrrr} & & 1 & 0 & -3 \\ & & \underline{1} & \underline{-1} & \underline{-4} \\ & & -4 & 1 & 2 \\ & -1 & 0 & 3 & \\ \underline{1} & \underline{0} & \underline{-3} & & \\ 1 & -1 & -7 & 4 & 2 \end{array}$$

This manuscript is dedicated to the memory of Celia LaGrange, a mathematics teacher for more than twenty-five years. She approached her teaching with enthusiasm and creativity and was an inspiration to her students.

We would also like to acknowledge the assistance of John Neff in the preparation of this manuscript.

TABLE 1
Multiplication Table

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	(1, 0)	(1, 2)	(1, 4)	(1, 6)	(1, 8)
3	3	6	9	(1, 2)	(1, 5)	(1, 8)	(2, 1)	(2, 4)	(2, 7)
4	4	8	(1, 2)	(1, 6)	(2, 0)	(2, 4)	(2, 8)	(3, 2)	(3, 6)
5	5	(1, 0)	(1, 5)	(2, 0)	(2, 5)	(3, 0)	(3, 5)	(4, 0)	(4, 5)
6	6	(1, 2)	(1, 8)	(2, 4)	(3, 0)	(3, 6)	(4, 2)	(4, 8)	(5, 4)
7	7	(1, 4)	(2, 1)	(2, 8)	(3, 5)	(4, 2)	(4, 9)	(5, 6)	(6, 3)
8	8	(1, 6)	(2, 4)	(3, 2)	(4, 0)	(4, 8)	(5, 6)	(6, 4)	(7, 2)
9	9	(1, 8)	(2, 7)	(3, 6)	(4, 5)	(5, 4)	(6, 3)	(7, 2)	(8, 1)

However, the digit 7 appearing here is not allowed in the new representation. But $-7 = -(1, -3) = (-1, 3)$, that is, $-7 = 3$ (carry "-1"). So the multiplication here should be the following:

$$\begin{array}{rrrrr} & & 1 & 0 & -3 \\ & & 1 & -1 & -4 \\ \hline & & -4 & 1 & 2 \\ & -1 & 0 & 3 & \\ 1 & 0 & -3 & & \\ \hline 1 & -2 & 3 & 4 & 2 \end{array}$$

This is the final answer in terms of our new representation. However, to pass back to the standard notation, we recall that $(1, -2) = 8$; so

$$(1, -2, 3, 4, 2) = 8342.$$

TABLE 2
Revised Multiplication Table

	1	2	3	4	5
1	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)
2	(0, 2)	(0, 4)	(1, −4)	(1, −2)	(1, 0)
3	(0, 3)	(1, −4)	(1, −1)	(1, 2)	(1, 5)
4	(0, 4)	(1, −2)	(1, 2)	(2, −4)	(2, 0)
5	(0, 5)	(1, 0)	(1, 5)	(2, 0)	(2, 5)

2. $499 \cdot 512$

$$499 = 500 - 1 = 5 \cdot 10^2 + (-1) \cdot 10^0 = (5, 0, -1);$$

$$512 = (5, 1, 2),$$

so $499 \cdot 512$ can be calculated as follows:

$$\begin{array}{rrrrrr} & & & 5 & 0 & -1 \\ & & & 5 & 1 & 2 \\ \hline & & 1 & 0 & 0 & -2 \\ & & 5 & 0 & -1 & \\ 2 & 5 & 0 & -5 & & \\ \hline 3 & -4 & -4 & -5 & -1 & -2 \end{array}$$

Starting from the fourth column from the right, we read this addition as follows: $1 + 5 = (1, -4)$, and so $1 + 5 = -4$ (carry "1"); $1 + 5 = (1, -4)$, that is, -4 (carry "1"); $1 + 2 = 3$. To transfer from $(3, -4, -4, -5, -1, -2)$ to the standard notation, the steps include the following:

$$\begin{aligned}(3, -4, -4, -5, -1, -2) &= (2, 6, -4, -5, -1, -2) \\ &= (2, 5, 6, -5, -1, -2) \\ &= (2, 5, 5, 5, -1, -2) \\ &= (2, 5, 5, 4, 9, -2) \\ &= (2, 5, 5, 4, 8, 8) \\ &= 255\,488.\end{aligned}$$

Here, $(3, -4) = 3 \cdot 10^1 - 4 \cdot 10^0 = 26$, $(6, -4) = 6 \cdot 10 - 4 = 56$, and so on.

Using the algorithm for the initial example, we have the following:

3. $739 \cdot 999$

$739 = (1, -3, 4, -1);$

$999 = (1, 0, 0, -1).$

$$\begin{array}{rrrrrrr} & & & 1 & -3 & 4 & -1 \\ & & & 1 & 0 & 0 & -1 \\ \hline & & & -1 & 3 & -4 & 1 \\ 1 & -3 & 4 & -1 & & & \\ \hline 1 & -3 & 4 & -2 & 3 & -4 & 1 \end{array} = 738\,261$$

(Continued on page 490)

I'm not going to take it anymore!" Well, there is a similar cry that we should all shout together—and that is this: "I AM A MATHEMATICS TEACHER AND I'M PROUD OF IT!" So let us recite this cry in unison, and let us make it loud enough to be heard not only throughout your school and throughout your city, but throughout the nation as well.

"I AM A MATHEMATICS TEACHER AND I AM PROUD OF IT!"

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This variation of the long multiplication algorithm is conceptually harder in that it involves negative numbers but is easier to perform, once mastered, since the size of the multiplication table required is smaller.

Mathematics Expressed in Trademarks

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