

A NEW BASIS FOR DISCRETE ANALYTIC POLYNOMIALS

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Abstract

A new basis $\{\pi_k(z)\}_{k=0}^{\infty}$ for discrete analytic polynomials is presented for which the series $\sum_{k=0}^{\infty} a_k \pi_k(z)$ converges absolutely to a discrete analytic function in the upper right quarter lattice whenever $\lim_{k \rightarrow \infty} |a_k|^{1/k} = 0$.

Introduction

Let Z be the group of integers and consider functions

$$f: Z \times Z \rightarrow \mathbf{C}$$

such that

$$(1.1) \quad f(x, y) + if(x + 1, y) - f(x + 1, y + 1) - if(x, y + 1) = 0$$

for every $(x, y) \in Z \times Z$. Such functions are termed *discrete entire*. If (1.1) holds only for $(x, y) \in G$, $G \subset Z \times Z$, then we say that f is *discrete analytic* in G .

Discrete analytic functions were introduced by Ferrand (1944) and the theory was developed by Duffin (1956) and others.

Duffin (1956) introduced the following basis for discrete analytic polynomials

$$(1.2) \quad \rho_k(z) = \frac{d^k}{dt^k} \left\{ \left(\frac{2+t}{2-t} \right)^x \left(\frac{2+it}{2-it} \right)^y \right\} \Big|_{t=0}$$

($z = x + iy$), which he called *pseudo-powers*.

Each $\rho_k(z)$ is a discrete entire function and a polynomial of degree k in (x, y) . Duffin (1956) showed that every discrete analytic polynomial can be expressed as a linear combination of these pseudo-powers.

Duffin and Peterson (1968) introduced an analogue of the McClaurin series in terms of these pseudo-powers. However, their analogue has the dis-

advantage that the convergence of $\sum_0^\infty a_n \xi^n$ on \mathbf{C} does not ensure the convergence of $\sum_0^\infty a_n \rho_n(z)$ on $Z \times Z$. In fact they showed that $\sum_0^\infty a_n \rho_n(z)$ converges on the whole lattice $Z \times Z$ only if

$$\overline{\lim} (|a_n| n!)^{1/n} < 2.$$

In Section 2 other “reasonable” bases for discrete analytic polynomials will be considered. These will be called *systems of pseudo-powers*, and it will be shown that the above drawback of Duffin’s basis $\{\rho_n(z)\}$ as regards the convergence of $\sum a_n \rho_n(z)$ cannot be removed by using other systems of pseudo-powers.

On the other hand, we shall construct a system of pseudo-powers $\{\pi_k(z)\}_0^\infty$ such that $\sum_0^\infty a_k \pi_k(z)$ converges absolutely on the quarter lattice $Z^+ \times Z^+ = \{(x + iy); x \text{ and } y \text{ integers, } x \geq 0, y \geq 0\}$ whenever $\sum_0^\infty a_k \xi^k$ converges on the entire plane. (The divergence of $\sum_0^\infty (2^n/n!) \rho_n(1, 0)$ shows that this property is not enjoyed by the Duffin–Peterson series.)

In Section 3 we shall consider the existence and uniqueness of the expansion $\sum_0^\infty a_k \pi_k(z)$. The discrete analogue of ‘multiplication by z ’ corresponding to the above basis will also be dealt with.

In Section 4, we discuss the lattice $Z_h^+ \times Z_h^+$ where $Z_h^+ = hZ^+$, $h > 0$ and show that if $\{\pi_k^h(z)\}_0^\infty$ is the corresponding basis then

$$\sum_0^\infty a_k \pi_k^h(z) \rightarrow \sum_0^\infty a_k z^k$$

when $h \downarrow 0$ along a sequence for which $z \in Z_h \times Z_h$, provided $\sum_0^\infty a_k \xi^k$ is an entire function of exponential type.

The analogous problem of representing monodiffic functions (that is functions satisfying

$$(i - 1)f(x, y) - if(x + 1, y) + f(x, y + 1) \equiv 0)$$

by a series of polynomials was considered by Atadžanov (1974).

2. The new basis

DEFINITION. A basis $\{p_n(z)\}_0^\infty$ for the discrete analytic polynomials is called a system of pseudo-powers if the following properties are satisfied:

(A1) $p_n(0) = 0$ for every $n > 0$

(A2) $\{p_n(z)\}$ satisfies the binomial identity

$$p_n(z_1 + z_2) = \sum_{k=0}^n \binom{n}{k} p_k(z_1) p_{n-k}(z_2)$$

(A3) $p_0 \equiv 1$ and for $n \geq 0$ $p_n(z) = z^n + \tilde{p}_{n-1}(x, y)$ where \tilde{p}_{n-1} is a polynomial of degree $\leq n - 1$.

It is readily checked that Duffin's basis $\{p_n(z)\}$ constitutes a system of pseudo-powers. On the other hand, Duffin's basis fails to satisfy the following:

$$(*) \quad \sum_0^\infty a_n p_n(z) \text{ converges absolutely for every } z \in Z \times Z$$

$$\text{if } \sum_0^\infty a_n \xi^n \text{ converges in the whole } \xi\text{-plane.}$$

One may ask: Does there exist a system of pseudo-powers satisfying (*)? That no such system exists follows from the next lemma.

LEMMA 2.1. *Let $\{p_k\}$ be any system of pseudo-powers. Then there exists a point z_0 in the half lattice $\{x + iy, y \geq 0\}$ and a complex number ζ_0 such that*

$$\sum_0^\infty \frac{\zeta_0^k}{k!} p_k(z_0) \text{ fails to converge absolutely.}$$

PROOF. Suppose that the statement is false, i.e., there exists a system of pseudo-powers $\{p_k\}$ such that

$$e(\zeta; z) = \sum_{k=0}^\infty \frac{\zeta^k}{k!} p_k(z)$$

converges absolutely for every point in the half lattice and for every complex number ζ . Then, for every such z , $e(\zeta, z)$ is an entire function in ζ and by (A2)

$$\begin{aligned} e(\zeta; z_1)e(\zeta; z_2) &= \left(\sum_0^\infty \frac{\zeta^k}{k!} p_k(z_1) \right) \left(\sum_0^\infty \frac{\zeta^r}{r!} p_r(z_2) \right) \\ &= \sum_0^\infty \frac{\zeta^n}{n!} \left[\sum_0^n \binom{n}{k} p_k(z_1) p_{n-k}(z_2) \right] \\ &= \sum_0^\infty \frac{\zeta^n}{n!} p_n(z_1 + z_2) = e(\zeta; z_1 + z_2). \end{aligned}$$

Thus $e(\zeta; x + iy) = f(\zeta)^x g(\zeta)^y$ where $f(\zeta) = e(\zeta; 1)$, $g(\zeta) = e(\zeta; i)$.

Since $e(\zeta; z)$ is discrete analytic in the upper half lattice (1.1) must be satisfied there:

$$f(\zeta)^x g(\zeta)^y \{1 + if(\zeta) - f(\zeta)g(\zeta) - ig(\zeta)\} = 0.$$

Thus

$$g(\zeta) = \frac{1 + if(\zeta)}{f(\zeta) + i}$$

and

$$e(\zeta; x + iy) = f(\zeta)^x \left(\frac{1 + if(\zeta)}{f(\zeta) + i} \right)^y.$$

Since $e(\zeta; z)$ is entire in ζ for each fixed z in the half lattice and in particular for $z = 1, -1, i$ we see that $f(\zeta), 1/f(\zeta)$ and $(1 + if(\zeta))/(f(\zeta) + i)$ are entire. But this implies that $f(\zeta)$ is entire and excludes the values 0 and $-i$. By the ‘‘little’’ Picard theorem [Rudin (1966), p. 324] this is too much to ask from a non-constant entire function. Evidently $f(\zeta)$ cannot be constant and so we arrive at a contradiction and the lemma is proved.

We saw that there is no system of pseudo-powers satisfying (*). The next theorem will demonstrate a system of pseudo-powers satisfying the following weaker property.

(A4) $\sum_0^\infty a_n p_n(z)$ converges absolutely for every

$$z \in Z^+ \times Z^+ = \{x + iy; x \geq 0, y \geq 0\} \quad \text{if} \quad \sum_0^\infty a_n \xi^n$$

converges in the whole ξ -plane.

The divergence of $\sum (2^n/n!) p_n(1)$ shows that Duffin’s basis does not satisfy (A4).

THEOREM 2.2. *The sequence of functions*

$$(2.1) \quad \pi_k(x, y) = \frac{d^k}{d\xi^k} \left\{ [(1 + i)e^{\xi/(1+i)} - i]^x [(1 - i)e^{-\xi/(1+i)} + i]^y \right\} \Big|_{\xi=0}$$

$k = 0, 1, 2, \dots$

constitutes a system of pseudo-powers satisfying (A4).

PROOF. The discrete analyticity of $\pi_k(x, y)$ is readily checked. (A1) is trivial, while (A2) follows from Leibnitz’ formula. Also, by a straightforward computation

$$(2.2) \quad \pi_{k+1}(x, y) = \frac{1}{1 + i} \{ (x - y)\pi_k(x, y) + ix\pi_k(x - 1, y) + iy\pi_k(x, y - 1) \}.$$

Since $\pi_0(x, y) \equiv 1$ it follows by induction that each $\pi_k(x, y)$ is a polynomial of degree k and that (A3) holds. Since Duffin (1956) showed that the dimension of the space of discrete analytic polynomials of degree $\leq k$ is $k + 1$, it follows that $\{\pi_r\}_0^k$ is a basis for the discrete analytic polynomials of degree $\leq k$ and consequently that $\{\pi_k\}_0^\infty$ is a basis for the discrete analytic polynomials. Thus $\{\pi_k\}$ is a system of pseudo-powers.

Now, let us note that for a fixed $z = x + iy \in Z^+ \times Z^+$

$$e(\zeta; x + iy) = \sum_0^\infty \pi_k(x, y) \frac{\zeta^k}{k!} = [(1 + i)e^{\zeta/(1+i)} - i]^x [(1 - i)e^{-\zeta/(1+i)} + i]^y.$$

Since x and y are non-negative integers, the right hand side is an entire function of exponential type and the Taylor coefficients being $\pi_k(x, y)/k!$ you have (Boas (1954), p. 11) that there exist constants C and T (depending on (x, y)) such that

$$|\pi_k(x, y)| \leq CT^k.$$

Thus $\sum_0^\infty a_k \pi_k(x, y)$ converges absolutely whenever $\overline{\lim} |a_k|^{1/k} = 0$, since $\sum_0^\infty a_k T^k$ does. This holds for every $(x, y) \in Z^+ \times Z^+$ and it follows that $\{\pi_k\}$ is a system of pseudo-powers satisfying (A4).

By Theorem (2.2) it follows that whenever $\sum_0^\infty a_k \xi^k$ is an entire function, i.e., whenever $\overline{\lim} |a_k|^{1/k} = 0$, then $\sum_0^\infty a_k \pi_k(z)$ converges to a discrete analytic function in $Z^+ \times Z^+$ (substitute in (1.1) and rearrange terms, using the fact that each $\pi_k(z)$ is discrete analytic).

Let \mathcal{A} be the algebra of entire functions and let \mathcal{D} be the set of discrete analytic functions on $Z^+ \times Z^+$. Define a mapping

$$T: \mathcal{A} \rightarrow \mathcal{D}$$

by

$$T\left(\sum_0^\infty a_n \xi^n\right) = \sum_0^\infty a_n \pi_n(z).$$

Let $\mathcal{F} \subset \mathcal{D}$ be the range of T . \mathcal{F} can be made into an algebra by requiring T to be a homomorphism:

$$\left(\sum_0^\infty a_k \pi_k(z)\right) \left(\sum_0^\infty b_k \pi_k(z)\right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k b_{n-k}\right) \pi_n(z).$$

Thus in our class \mathcal{F} , multiplication is defined for every pair $f, g \in \mathcal{F}$. This is an improvement on the multiplication in the Duffin–Peterson class,

$$\mathcal{F}_{DP} = \left\{ \sum_0^\infty a_n \rho_n(z); \overline{\lim} (|a_n| n!)^{1/n} < 2 \right\}$$

which is only defined on a subset of $\mathcal{F}_{DP} \times \mathcal{F}_{DP}$. In particular $\exp f$ is well defined in our class:

$$\exp\left(\sum_0^\infty a_k \pi_k(z)\right) = T\left(\exp\left(\sum_0^\infty a_k \xi^k\right)\right).$$

3. Existence and uniqueness of Taylor expansion

Formula (2.2) motivates the following analogue for the continuous “multiplication by z ”

$$(3.1) \quad \mathfrak{z}f(x, y) = \frac{1}{1+i} \{ (x-y)f(x, y) + xf(x-1, y) + iyf(x, y-1) \}.$$

It is readily checked that if f is discrete analytic, then so is $\mathfrak{z}f$ and, by (2.2)

$$\mathfrak{z}\pi_k = \pi_{k+1} \quad \mathfrak{z}e(\xi; x+iy) = \frac{d}{d\xi} e(\xi; x+iy).$$

Let us restrict attention to \mathcal{D} , the class of discrete analytic functions on $Z^+ \times Z^+$. It was shown in Zeilberger (to appear) that each $f \in \mathcal{D}$ is uniquely determined by the pair of formal power series (ϕ_f, ψ_f) where

$$\phi_f(X) = \sum_{x=0}^{\infty} f(x, 0)X^x, \quad \psi_f(Y) = \sum_{y=0}^{\infty} f(0, y)Y^y,$$

and we write $f = (\phi_f, \psi_f)$.

Since $\mathfrak{z}f(x, 0) = 1/(1+i)\{xf(x, 0) + ix f(x-1, 0)\}$

$$\begin{aligned} \sum_{x=0}^{\infty} \mathfrak{z}f(x, 0)X^x &= \frac{1}{1+i} \sum_{x=0}^{\infty} x(f(x, 0) + if(x-1, 0))X^x \\ &= \frac{X}{1+i} \frac{d}{dX} [(1+iX)\phi_f(X)]. \end{aligned}$$

Similarly

$$\sum_{y=0}^{\infty} \mathfrak{z}f(0, y)Y^y = \frac{Y}{1+i} \frac{d}{dY} [(iY-1)\psi_f(Y)].$$

So the operation of \mathfrak{z} in terms of formal power series is

$$(3.2) (\phi_f, \psi_f) \rightarrow \frac{1}{1+i} \left(X \frac{d}{dX} [(1+iX)\phi_f(X)], \quad Y \frac{d}{dY} [(iY-1)\psi_f(Y)] \right).$$

Thus $\mathfrak{z}f \equiv 0$ iff

$$\phi_f(X) = \frac{C}{1+iX}; \quad \psi_f(Y) = \frac{C}{1-iY}.$$

(The constants agree since $\phi_f(0) = f(0, 0) = \psi_f(0)$.) So, unfortunately, \mathfrak{z} has a non-trivial kernel.

Clearly, $\mathfrak{z}f(0) = 0$ for every function f discrete analytic in $Z^+ \times Z^+$. Let $g \in \mathcal{D}$, $g(0) = 0$ then $f \in \mathcal{D}$ given by

$$\begin{aligned} \phi_f(X) &= \frac{1+i}{1+iX} \int \frac{\phi_g(X)}{X} dX = \frac{1+i}{1+iX} \left[\sum_1^{\infty} \frac{g(x,0)X^x}{x} + C \right] \\ \psi_f(Y) &= \frac{1+i}{1-iY} \int \frac{\phi_g(Y)}{Y} dY = \frac{1+i}{1-iY} \left[\sum_1^{\infty} \frac{g(0,y)Y^y}{y} + C \right] \end{aligned}$$

solves $\mathfrak{z}f = g$.

We have thus obtained

THEOREM 3.1. *The operator*

$$\mathfrak{z} : \mathcal{D} \rightarrow \mathcal{D}$$

has range $\{f \in \mathcal{D}; f(0,0) = 0\}$ and kernel $\{Cf_0\}$ where $f_0 \in \mathcal{D}$ is given by

$$\phi_{f_0} = \frac{1}{1+iX}; \quad \psi_{f_0} = \frac{1}{1-iY}.$$

Let us consider the class $\mathcal{F} \subset \mathcal{D}$ defined at the end of Section 2. It is not yet known whether the inclusion $\mathcal{F} \subset \mathcal{D}$ is proper or not; i.e., whether every discrete analytic function on $Z^+ \times Z^+$ possesses a discrete Taylor expansion

$$(3.3) \quad f(z) = \sum_{k=0}^{\infty} a_k \pi_k(z).$$

Theorem (3.1) implies that even if such a representation exists it need not be unique. However if attention is restricted to the class

$$\mathcal{F}_\epsilon = \left\{ \sum_0^{\infty} a_k \pi_k(z); \quad \overline{\lim} (k! |a_k|)^{1/k} < \infty \right\}$$

then the representation (3.3) is unique, as follows from the following

THEOREM 3.2. *If $\sum_0^{\infty} a_k \pi_k(z) \equiv \tilde{0}$ in $Z^+ \times Z^+$ and $\overline{\lim} (k! |a_k|)^{1/k} < \infty$ then $a_k = 0$ for every k .*

PROOF. By definition (2.1)

$$\pi_k(x, y) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{[(1+i)e^{\zeta/(1+i)} - i]^x [(1-i)e^{-\zeta/(1+i)} + i]^y}{\zeta^{k+1}} d\zeta$$

where Γ is any contour surrounding 0. So,

$$\begin{aligned} f(z) &= \sum_0^{\infty} a_k \pi_k(x, y) \\ &= \frac{1}{2\pi i} \int_{\Gamma} \left(\sum \frac{k! a_k}{\zeta^{k+1}} \right) [(1+i)e^{\zeta/(1+i)} - i]^x [(1-i)e^{-\zeta/(1+i)} + i]^y d\zeta \end{aligned}$$

for any contour Γ for which

$$f_B(\zeta) = \sum_{k=0}^{\infty} \frac{k! a_k}{\zeta^{k+1}}$$

is defined. $f_B(\zeta)$ is the Borel transform of

$$f^c(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$$

and $f_B(\zeta)$ converges for $|\zeta| \geq \text{type } f^c$ (see Boas (1954), p. 73). Thus

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f_B(\zeta) [(1+i)e^{\zeta/(1+i)} - i]^x [(1-i)e^{-\zeta/(1+i)} + i]^y d\zeta$$

and for some constant M

$$|f(x, 0)| \leq CM^x$$

and $\phi_f(t) = \sum_{x=0}^{\infty} f(x, 0)t^x$ converges in the disc $|t| < 1/M$. We have then

$$\begin{aligned} \phi_f(t) &= \sum_{x=0}^{\infty} f(x, 0)t^x = \frac{1}{2\pi i} \int_{\Gamma} f_B(\zeta) \sum_0^{\infty} [(1+i)e^{\zeta/(1+i)} - i]^x t^x \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f_B(\zeta) d\zeta}{1 - [(1+i)e^{\zeta/(1+i)} - i]t}. \end{aligned}$$

The right hand side defines an analytic function in any region in the t -plane for which the denominator of the integrand does not vanish in a neighborhood of Γ in the ζ -plane. In particular, this includes a neighborhood of the point i in the t -plane. Thus for any discrete analytic function of class \mathcal{F}_e

$$\phi_f(t) = \sum_{x=0}^{\infty} f(x, 0)t^x$$

whose radius of convergence is in general smaller than 1, can be analytically continued through the boundary of the circle of convergence to a neighborhood of $t = i$.

Now $\sum_0^{\infty} a_k \pi_k(z) \equiv 0$ implies $a_0 = 0$ and

$$\mathfrak{z} \left(\sum_1^{\infty} a_k \pi_{k-1}(z) \right) = 0.$$

Let $g_1(z) = \sum_1^{\infty} a_k \pi_{k-1}(z)$. Then $g_1 \in \mathcal{F}_e$ and hence $\phi_{g_1}(t)$ can be analytically continued to a neighborhood of $t = i$. But $\mathfrak{z}g_1 \equiv 0$ implies, by Theorem 3.1, that $\phi_{g_1}(t) = C/(1+it)$ for some constant C . This forces $C = 0$ for, otherwise ϕ_{g_1} would have a pole at $t = i$. Thus,

$$g_1(z) = \sum_1^{\infty} a_k \pi_{k-1}(z) \equiv 0 \quad \text{and} \quad a_1 = 0.$$

Continuing inductively we get that $a_k = 0$ for every k and the theorem is proved.

4. Limiting behavior as $h \downarrow 0$

Let $h > 0$. For the lattice of mesh size h

$$Z_h \times Z_h = \{(hm, hn); m, n \in Z\}$$

discrete analyticity is defined by

$$(4.1) \quad F(x, y) + iF(x + h, y) - F(x + h, y + h) - iF(x, y + h) = 0.$$

The above discussion carries over to discrete analytic functions for such lattices (all it amounts to is a change of scale). Now we have the basis

$$(4.2) \quad \pi_k^h(x, y) = \frac{d^k}{d\xi^k} \left\{ [(1+i)e^{\xi h/(1+i)} - i]^{x/h} [(1-i)e^{-\xi h/(1+i)} + i]^{y/h} \right\} \Big|_{\xi=0}.$$

And for discrete analytic functions on the lattice $Z_h \times Z_h$ the exponential function is

$$e_h(x, y) = \sum_{k=0}^{\infty} \pi_k^h(x, y) \frac{\xi^k}{k!} = [(1+i)e^{\xi h/(1+i)} - i]^{x/h} [(1-i)e^{-\xi h/(1+i)} + i]^{y/h}.$$

Now as $h \downarrow 0$

$$[(1+i)e^{\xi h/(1+i)} - i]^{1/h} \rightarrow e^\xi \quad [(1-i)e^{-\xi h/(1+i)} + i]^{1/h} \rightarrow e^{i\xi}.$$

So $e_h(x, y) \rightarrow e^{\xi(x+iy)}$ and consequently

$$\pi_k^h(z) \rightarrow z^k \quad \text{as } h \downarrow 0.$$

Suppose $|a_n| \leq C \zeta_0^n / n!$ for some constants C and ζ_0 , by dominated convergence

$$f^h(z) = \sum_0^\infty a_k \pi_k^h(z) \rightarrow \sum_0^\infty a_k z^k$$

as $h \downarrow 0$. We obtained

LEMMA 4.1. *If $\overline{\lim} (|a_k| k!)^{1/k} < \infty$ then $f^h(z) \rightarrow f^c(z) = \sum_0^\infty a_k z^k$ along a sequence $h \downarrow 0$ for which $z \in Z_h^+ \times Z_h^+$.*

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References

- B. A. Atadzanov (1974), 'Representation of analytic functions of a discrete complex variable by means of basis elements'. *Metričeskie Voprosy Teor. Funkcij, Otobražaniu* 5, 13–20 (Russian).
- R. P. Boas, Jr (1954), *Entire Functions* (Academic Press, New York).
- R. J. Duffin (1956), 'Basic properties of discrete analytic functions', *Duke Math. J.* 23, 335–363.
- R. J. Duffin and Elmore L. Peterson (1968), 'The discrete analogue of a class of entire functions'. *J. Math. Anal. Appl.* 21, 619–642.
- J. Ferrand (1944), 'Fonctions préharmonique et fonctions préholomorphes', *Bull. Sci. Math.* 68, 152–180.
- W. Rudin (1966), *Real and Complex Analysis* (McGraw-Hill, New York).
- D. Zeilberger (to appear), 'A new approach to the theory of discrete analytic functions'. *J. Math. Anal. Appl.*

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