A note on Kyle Petersen's conjecture ([3])

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It is well known (cf. [1]) that the Fibonacci polynomials $F_n(x,s)=\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-j}{j} s^j x^{n-1-2j}$ satisfy

$$F_n(x+y,-xy)=\frac{x^n-y^n}{x-y}$$

and the Lucas polynomials $L_n(x,s)=\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-j} \binom{n-j}{j} s^j x^{n-2j}$ with $L_0(x,s)=2$ satisy

$$L_n(x+y,-xy)=x^n+y^n.$$

Consider the Kyle map $K\left(\sum_j (-1)^j \gamma_j q^j (1+q)^{n-2j}\right)=\sum_j \gamma_j$ which (as Richard Stanley [2] observed) satisfies $K(f(q)g(q))=K(f(q))K(g(q))$.

Since $K\left(\frac{1-q^n}{1-q}\right)=K(F_n(1+q,-q))=F_n(1,1)=F_n$ the above mentioned result

$$K\left([n]!\right)=\prod_{j=1}^n F_j$$ follows.

In the same way $K\left(1+q^n\right)=K\left(L_n(1+q,-q)\right)=L_n(1,1)=L_n$ where $\left(L_n\right)=\left(2,1,3,4,7,11,18,\cdots\right)$ is the sequence of Lucas numbers.

Thus $K\left(\prod_{j=1}^n (1+q^j)\right)=\prod_{j=1}^n L_j$.

References

[1] H. W. Gould, A Fibonacci formula of Lucas and its subsequent manifestations and rediscoveries, Fib. Quart. 15 (1977), 25-29

[2] R. Stanley, Email message ,
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/kylefb.html

[3] D. Zeilberger, Proof of Kyle Petersen's amazing conjecture relating the q- and Fibonacci analogs of n!, http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/kylefb.html