AUTOMATIC PROOF OF THE CONGRUENCES

Congruence 1

Let $r := (r_1) = (-3)$. We apply Lemma 4.4 in [1] with $(m, M, N, t, (r_\delta)) := (11, 1, 11, 7, r)$ and with $(a_\delta) = a$ where $a := (a_1, a_{11}) = (34, -3)$. We write a representative $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the double coset $T := \Gamma_0(N)A\Gamma_\infty$ as (a, c) because for fixed a, c any value of b, d such that $A \in \mathrm{SL}_2(\mathbb{Z})$ gives an element in the double coset T. Then a complete set of representatives is given by:

$$(1, 0), (0, 1).$$

We have that

$$P(t) = \{7\}.$$

Using this information we can now compute the bound $\lfloor \nu \rfloor = 13$ of Lemma 4.4 in [1]. Recall that

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta | M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}.$$

Then by Lemma 4.4 in [1] we have that for any positive integer u:

$$a(11n + 7) \equiv 0 \pmod{u}, \text{ for all } n \in \mathbb{N}$$

iff

$$a(11n + 7) \equiv 0 \pmod{u}, \text{ for } n = 0, \ldots, 13\ .$$

Congruence 2

Let $r := (r_1) = (-3)$. We apply Lemma 4.4 in [1] with $(m, M, N, t, (r_\delta)) := (17, 1, 17, 15, r)$ and with $(a_\delta) = a$ where $a := (a_1, a_{17}) = (52, -3)$. We write a representative $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the double coset $T := \Gamma_0(N)A\Gamma_\infty$ as (a, c) because for fixed a, c any value of b, d such that $A \in \mathrm{SL}_2(\mathbb{Z})$ gives an element in the double coset T. Then a complete set of representatives is given by:

$$(1, 0), (0, 1).$$

We have that

$$P(t) = \{15\}.$$

Using this information we can now compute the bound $\lfloor \nu \rfloor = 33$ of Lemma 4.4 in [1]. Recall that

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta | M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}.$$

Then by Lemma 4.4 in [1] we have that for any positive integer u:

$$a(17n + 15) \equiv 0 \pmod{u}, \text{ for all } n \in \mathbb{N}$$

iff

$$a(17n + 15) \equiv 0 \pmod{u}, \text{ for } n = 0, \ldots, 33\ .$$

Congruence 3

Let $r := (r_1) = (-7)$. We apply Lemma 4.4 in [1] with $(m, M, N, t, (r_\delta)) := (19, 1, 19, 9, r)$ and with $(a_\delta) = a$ where $a := (a_1, a_{19}) = (134, -7)$. We write a representative $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the double coset $T := \Gamma_0(N)A\Gamma_\infty$ as (a, c) because for fixed a, c any value of b, d such that $A \in \mathrm{SL}_2(\mathbb{Z})$ gives an element in the double coset T. Then a complete set of representatives is given by:

$$(1, 0), (0, 1).$$

We have that

$$P(t) = \{9\}.$$

Using this information we can now compute the bound $\lfloor \nu \rfloor = 99$ of Lemma 4.4 in [1]. Recall that

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta | M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}.$$

Then by Lemma 4.4 in [1] we have that for any positive integer u:

$$a(19n + 9) \equiv 0 \pmod{u}, \text{ for all } n \in \mathbb{N}$$

iff

$$a(19n + 9) \equiv 0 \pmod{u}, \text{ for } n = 0, \ldots, 99 \text{ .}$$

Congruence 4

Let $r := (r_1) = (-9)$. We apply Lemma 4.4 in [1] with $(m, M, N, t, (r_\delta)) := (19, 1, 19, 17, r)$ and with $(a_\delta) = a$ where $a := (a_1, a_{19}) = (172, -9)$. We write a representative $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the double coset $T := \Gamma_0(N)A\Gamma_\infty$ as (a, c) because for fixed a, c any value of b, d such that $A \in \mathrm{SL}_2(\mathbb{Z})$ gives an element in the double coset T. Then a complete set of representatives is given by:

$$(1, 0), (0, 1).$$

We have that

$$P(t) = \{17\}.$$

Using this information we can now compute the bound $\lfloor \nu \rfloor = 127$ of Lemma 4.4 in [1]. Recall that

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta | M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}.$$

Then by Lemma 4.4 in [1] we have that for any positive integer u:

$$a(19n + 17) \equiv 0 \pmod{u}, \text{ for all } n \in \mathbb{N}$$

iff

$$a(19n + 17) \equiv 0 \pmod{u}, \text{ for } n = 0, \ldots, 127 \text{ .}$$

References

[1] S. Radu. An Algorithmic Approach to Ramanujan's Congruences. *Ramanujan Journal*, 20:215–251, 2009.