In[∘]:=

```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type ?HolonomicFunctions for help.
```

Show that DxH satisfies the equation for DxR

In[∘]:=

```
(* This is the identity we wish to show,
where Dp(x) is defined by the second-order recurrence below. *)
TraditionalForm[
 Dp[x] == R1[p] + Sum[R2[i, p], {i, 1, p - 1}] + Sum[R3[i, p] * Di[x], {i, 1, p - 1}]]
```

Out[∘]//TraditionalForm=

$$D_p(x) = \sum_{i=1}^{p-1} D_i(x)\,\mathrm{R3}(i,\,p) + \sum_{i=1}^{p-1} \mathrm{R2}(i,\,p) + \mathrm{R1}(p)$$

In[∘]:=
```
(* This is the recursive definition of D_p(x) *)
R1[p_] := (-12 * p^2 * (x - p) / (x^3 - x + p - p^3)) * (-28 / 9 * x^2 + 29 / 45 + 274 / 45 * p^2 -
    1 / 6 * (x^3 - x) / p + 1 / 5 * (x^3 - x) * (x + p) / (x^2 + x * p + p^2 - 1) - 13 / 9 * x * p);
R2[i_, p_] := (12 * i^2 * (i - x) / x / (x + 1) / (x - 1)) *
   ((-12) * p * (p - i) * (x - p) / (x - i) / (x^3 - x + p - p^3)) *
   (5 / 18 * (x^3 - x) * 1 / p + 38 / 15 * p^2 + (x^3 - x) / 5 * (x + p) / (x^2 + p^2 + x * p - 1) -
    13 / 9 * i * p - 13 / 9 * (p - i) * x + 49 / 45);
R3[i_, p_] := (-12) * p * (p - i) * (x - p) / (x - i) / (x^3 - x + p - p^3);
Clear[DxR];
DxR[p_, x_] := DxR[p, x] =
   If[p == 1,
    Together[(-12 * (x - 1) / (x^3 - x) *
       (-28 / 9 * x^2 + 29 / 45 + 274 / 45 - 1 / 6 * (x^3 - x) + 1 / 5 * (x^2 - 1) - 13 / 9 * x))],
    Together[R1[p] + Sum[R2[i, p], {i, 1, p - 1}] +
      Sum[R3[i, p] * DxR[i, x], {i, 1, p - 1}]]
   ];
Table[DxR[n, x], {n, 6}]
```

Out[∘]=

$$\left\{\frac{2\left(-588+115x+262x^2+15x^3\right)}{15x(1+x)},\right.$$

$$\frac{4\left(16056+1266x-3649x^2-4910x^3+490x^4+524x^5+15x^6\right)}{15(-1+x)x(1+x)\left(3+2x+x^2\right)},$$

$$\left(6\left(-372672-48120x+44530x^2+112525x^3-1642x^4-8625x^5-5422x^6+375x^7+262x^8+5x^9\right)\right)\Big/\left(5(-2+x)(-1+x)x(1+x)\left(3+2x+x^2\right)\left(8+3x+x^2\right)\right),$$

$$\left(8\left(149921280+22889088x-10710360x^2-43049898x^3-828625x^4+2305958x^5+2582700x^6-79350x^7-100470x^8-38046x^9+2020x^{10}+1048x^{11}+15x^{12}\right)\right)\Big/\left(15(-3+x)(-2+x)(-1+x)x(1+x)\left(3+2x+x^2\right)\left(8+3x+x^2\right)\left(15+4x+x^2\right)\right),$$

$$\left(10\left(-6830438400-1146908160x+315279648x^2+1890217728x^3+73433634x^4-73063357x^5-120642254x^6+1284739x^7+3665192x^8+2446002x^9-80136x^{10}-57590x^{11}-14746x^{12}+635x^{13}+262x^{14}+3x^{15}\right)\right)\Big/\left(3(-4+x)(-3+x)(-2+x)(-1+x)x(1+x)\left(3+2x+x^2\right)\left(8+3x+x^2\right)\left(15+4x+x^2\right)\left(24+5x+x^2\right)\right),$$

$$\left.\left(12\left(4066288128000+727574400000x-126670875840x^2-1097400884256x^3-57311836140x^4+31601361388x^5+71558841485x^6+176507096x^7-1712965710x^8-1638575932x^9+33145995x^{10}+32005056x^{11}+14444480x^{12}-451580x^{13}-227805x^{14}-42296x^{15}+1530x^{16}+524x^{17}+5x^{18}\right)\right)\Big/\left(5(-5+x)(-4+x)(-3+x)(-2+x)(-1+x)x(1+x)\left(3+2x+x^2\right)\left(8+3x+x^2\right)\left(15+4x+x^2\right)\left(24+5x+x^2\right)\left(35+6x+x^2\right)\right)\right\}$$

In[∘]:=
```
Clear[DxH];
DxH[n_, x_] := DxH[n, x] =
   Which[n == 1, 2 / 15 * (15 * x^3 + 262 * x^2 + 115 * x - 588) / x / (x + 1),
    n == 2,
```

```
4 / 15 / x * (15 * x ^ 6 + 524 * x ^ 5 + 490 * x ^ 4 - 4910 * x ^ 3 - 3649 * x ^ 2 + 1266 * x + 16 056) /
    (x + 1) / (x - 1) / (x ^ 2 + 2 * x + 3),
True,
Together[
  (* This is the conjectured recurrence that D_p(x) appears to satisfy. *)
  (2 * (100 * n ^ 12 - 26 * n ^ 11 * x - 351 * n ^ 9 * x ^ 3 + 78 * n ^ 8 * x ^ 4 + 453 * n ^ 6 * x ^ 6 -
        78 * n ^ 5 * x ^ 7 - 199 * n ^ 3 * x ^ 9 + 26 * n ^ 2 * x ^ 10 - 3 * x ^ 12 - 1200 * n ^ 11 +
        286 * n ^ 10 * x + 3159 * n ^ 8 * x ^ 3 - 624 * n ^ 7 * x ^ 4 - 2718 * n ^ 5 * x ^ 6 +
        390 * n ^ 4 * x ^ 7 + 597 * n ^ 2 * x ^ 9 - 52 * n * x ^ 10 + 5900 * n ^ 10 -
        897 * n ^ 9 * x - 78 * n ^ 8 * x ^ 2 - 11 730 * n ^ 7 * x ^ 3 + 1122 * n ^ 6 * x ^ 4 +
        156 * n ^ 5 * x ^ 5 + 6642 * n ^ 4 * x ^ 6 - 183 * n ^ 3 * x ^ 7 - 78 * n ^ 2 * x ^ 8 -
        380 * n * x ^ 9 + 12 * x ^ 10 - 15 000 * n ^ 9 - 507 * n ^ 8 * x + 624 * n ^ 7 * x ^ 2 +
        23 142 * n ^ 6 * x ^ 3 + 2004 * n ^ 5 * x ^ 4 - 780 * n ^ 4 * x ^ 5 - 8448 * n ^ 3 * x ^ 6 -
        1011 * n ^ 2 * x ^ 7 + 156 * n * x ^ 8 - 18 * x ^ 9 + 19 500 * n ^ 8 + 9312 * n ^ 7 * x -
        1575 * n ^ 6 * x ^ 2 - 26 037 * n ^ 5 * x ^ 3 - 10 086 * n ^ 4 * x ^ 4 + 963 * n ^ 3 * x ^ 5 +
        5655 * n ^ 2 * x ^ 6 + 828 * n * x ^ 7 - 18 * x ^ 8 - 7200 * n ^ 7 - 23 688 * n ^ 6 * x +
        714 * n ^ 5 * x ^ 2 + 16 701 * n ^ 4 * x ^ 3 + 15 336 * n ^ 3 * x ^ 4 + 231 * n ^ 2 * x ^ 5 -
        1662 * n * x ^ 6 + 54 * x ^ 7 - 13 900 * n ^ 6 + 29 027 * n ^ 5 * x + 3444 * n ^ 4 * x ^ 2 -
        5741 * n ^ 3 * x ^ 3 - 10 868 * n ^ 2 * x ^ 4 - 516 * n * x ^ 5 + 12 * x ^ 6 + 21 000 * n ^ 5 -
        18 703 * n ^ 4 * x - 6888 * n ^ 3 * x ^ 2 + 771 * n ^ 2 * x ^ 3 + 3064 * n * x ^ 4 -
        54 * x ^ 5 - 11 600 * n ^ 4 + 5784 * n ^ 3 * x + 5265 * n ^ 2 * x ^ 2 + 68 * n * x ^ 3 -
        3 * x ^ 4 + 2400 * n ^ 3 - 588 * n ^ 2 * x - 1506 * n * x ^ 2 + 18 * x ^ 3) *
     n / (n - 1) / (n - 1 - x) / (n ^ 2 + n * x + x ^ 2 - 1) /
      (100 * n ^ 9 - 26 * n ^ 8 * x - 251 * n ^ 6 * x ^ 3 + 52 * n ^ 5 * x ^ 4 + 202 * n ^ 3 * x ^ 6 -
        26 * n ^ 2 * x ^ 7 + 3 * x ^ 9 - 1350 * n ^ 8 + 312 * n ^ 7 * x + 2259 * n ^ 5 * x ^ 3 -
        390 * n ^ 4 * x ^ 4 - 909 * n ^ 2 * x ^ 6 + 78 * n * x ^ 7 + 7800 * n ^ 7 - 1309 * n ^ 6 * x -
        52 * n ^ 5 * x ^ 2 - 8231 * n ^ 4 * x ^ 3 + 740 * n ^ 3 * x ^ 4 + 52 * n ^ 2 * x ^ 5 +
        1313 * n * x ^ 6 - 61 * x ^ 7 - 25 200 * n ^ 6 + 1953 * n ^ 5 * x + 390 * n ^ 4 * x ^ 2 +
        15 501 * n ^ 3 * x ^ 3 + 180 * n ^ 2 * x ^ 4 - 156 * n * x ^ 5 - 606 * x ^ 6 + 49 800 * n ^ 5 +
        1601 * n ^ 4 * x - 942 * n ^ 3 * x ^ 2 - 15 916 * n ^ 2 * x ^ 3 - 1482 * n * x ^ 4 +
        113 * x ^ 5 - 61 650 * n ^ 4 - 9417 * n ^ 3 * x + 729 * n ^ 2 * x ^ 2 + 8490 * n * x ^ 3 +
        900 * x ^ 4 + 46 700 * n ^ 3 + 12 874 * n ^ 2 * x + 169 * n * x ^ 2 - 1855 * x ^ 3 -
        19 800 * n ^ 2 - 7788 * n * x - 294 * x ^ 2 + 3600 * n + 1800 * x) * DxH[n - 1, x] -
    (n ^ 2 + n * x + x ^ 2 - 4 * n - 2 * x + 3) * (100 * n ^ 9 - 26 * n ^ 8 * x - 251 * n ^ 6 * x ^ 3 +
        52 * n ^ 5 * x ^ 4 + 202 * n ^ 3 * x ^ 6 - 26 * n ^ 2 * x ^ 7 + 3 * x ^ 9 - 450 * n ^ 8 +
        104 * n ^ 7 * x + 753 * n ^ 5 * x ^ 3 - 130 * n ^ 4 * x ^ 4 - 303 * n ^ 2 * x ^ 6 + 26 * n * x ^ 7 +
        600 * n ^ 7 + 147 * n ^ 6 * x - 52 * n ^ 5 * x ^ 2 - 701 * n ^ 4 * x ^ 3 - 300 * n ^ 3 * x ^ 4 +
        52 * n ^ 2 * x ^ 5 + 101 * n * x ^ 6 - 9 * x ^ 7 - 805 * n ^ 5 * x + 130 * n ^ 4 * x ^ 2 +
        147 * n ^ 3 * x ^ 3 + 580 * n ^ 2 * x ^ 4 - 52 * n * x ^ 5 - 600 * n ^ 5 + 831 * n ^ 4 * x +
        98 * n ^ 3 * x ^ 2 + 26 * n ^ 2 * x ^ 3 - 202 * n * x ^ 4 + 9 * x ^ 5 + 450 * n ^ 4 - 199 * n ^ 3 * x -
        277 * n ^ 2 * x ^ 2 + 26 * n * x ^ 3 - 100 * n ^ 3 - 52 * n ^ 2 * x + 101 * n * x ^ 2 - 3 * x ^ 3) *
     n / (n - 2) / (100 * n ^ 9 - 26 * n ^ 8 * x - 251 * n ^ 6 * x ^ 3 + 52 * n ^ 5 * x ^ 4 +
         202 * n ^ 3 * x ^ 6 - 26 * n ^ 2 * x ^ 7 + 3 * x ^ 9 - 1350 * n ^ 8 + 312 * n ^ 7 * x +
```

```
            2259 * n^5 * x^3 - 390 * n^4 * x^4 - 909 * n^2 * x^6 + 78 * n * x^7 + 7800 * n^7 -
            1309 * n^6 * x - 52 * n^5 * x^2 - 8231 * n^4 * x^3 + 740 * n^3 * x^4 +
            52 * n^2 * x^5 + 1313 * n * x^6 - 61 * x^7 - 25 200 * n^6 + 1953 * n^5 * x +
            390 * n^4 * x^2 + 15 501 * n^3 * x^3 + 180 * n^2 * x^4 - 156 * n * x^5 -
            606 * x^6 + 49 800 * n^5 + 1601 * n^4 * x - 942 * n^3 * x^2 - 15 916 * n^2 * x^3 -
            1482 * n * x^4 + 113 * x^5 - 61 650 * n^4 - 9417 * n^3 * x + 729 * n^2 * x^2 +
            8490 * n * x^3 + 900 * x^4 + 46 700 * n^3 + 12 874 * n^2 * x + 169 * n * x^2 -
            1855 * x^3 - 19 800 * n^2 - 7788 * n * x - 294 * x^2 + 3600 * n + 1800 * x) /
          (n^2 + n * x + x^2 - 1) * DxH[n - 2, x])]];
(* Check that the two sequences, the original one and the one
 defined by the recurrence, are the same: *)
Table[Together[DxH[n, x] - DxR[n, x]], {n, 6}]
```

Out[•]= {0, 0, 0, 0, 0, 0}

In[•]:= `annR1 = Annihilator[R1[p], S[p]]`

Out[•]= $\{(-232 p^4 - 232 p^5 - 2018 p^6 - 1960 p^7 + 1702 p^8 + 2192 p^9 + 548 p^{10} - 292 p^3 x - 408 p^4 x - 1351 p^5 x - 2200 p^6 x + 3069 p^7 x + 4960 p^8 x + 1514 p^9 x - 118 p^2 x^2 - 438 p^3 x^2 + 1386 p^4 x^2 - 490 p^5 x^2 + 2007 p^6 x^2 + 6348 p^7 x^2 + 2618 p^8 x^2 - 15 p x^3 - 206 p^2 x^3 + 1484 p^3 x^3 + 2260 p^4 x^3 - 383 p^5 x^3 + 2804 p^6 x^3 + 2219 p^7 x^3 - 30 p x^4 + 367 p^2 x^4 + 2560 p^3 x^4 - 1653 p^4 x^4 - 1886 p^5 x^4 + 707 p^6 x^4 + 15 p x^5 + 850 p^2 x^5 - 630 p^3 x^5 - 3888 p^4 x^5 - 1096 p^5 x^5 + 60 p x^6 + 58 p^2 x^6 - 2566 p^3 x^6 - 1537 p^4 x^6 + 15 p x^7 - 644 p^2 x^7 - 1006 p^3 x^7 - 30 p x^8 - 307 p^2 x^8 - 15 p x^9) S_p + (-1212 p - 5222 p^2 - 6576 p^3 + 2714 p^4 + 13500 p^5 + 9690 p^6 - 2424 p^7 - 6634 p^8 - 3288 p^9 - 548 p^{10} - 588 x - 2630 p x - 1135 p^2 x + 11339 p^3 x + 22368 p^4 x + 10698 p^5 x - 11979 p^6 x - 17893 p^7 x - 8666 p^8 x - 1514 p^9 x - 473 x^2 + 2378 p x^2 + 15487 p^2 x^2 + 24550 p^3 x^2 + 4979 p^4 x^2 - 25832 p^5 x^2 - 30875 p^6 x^2 - 14596 p^7 x^2 - 2618 p^8 x^2 + 1553 x^3 + 7397 p x^3 + 8193 p^2 x^3 - 10199 p^3 x^3 - 31430 p^4 x^3 - 29392 p^5 x^3 - 12729 p^6 x^3 - 2219 p^7 x^3 + 1223 x^4 - 144 p x^4 - 12234 p^2 x^4 - 23828 p^3 x^4 - 18382 p^4 x^4 - 6128 p^5 x^4 - 707 p^6 x^4 - 1327 x^5 - 6497 p x^5 - 9628 p^2 x^5 - 3962 p^3 x^5 + 1592 p^4 x^5 + 1096 p^5 x^5 - 1027 x^6 - 1606 p x^6 + 1466 p^2 x^6 + 3582 p^3 x^6 + 1537 p^4 x^6 + 347 x^7 + 1715 p x^7 + 2374 p^2 x^7 + 1006 p^3 x^7 + 277 x^8 + 584 p x^8 + 307 p^2 x^8 + 15 x^9 + 15 p x^9)\}$

In[∘]:=
```
annSum1 = Annihilator[Sum[R2[i, p], {i, 1, p - 1}], S[p]]
```

Out[∘]=

$$
\begin{aligned}
&\{(600\,p^4 - 450\,p^6 - 150\,p^7 - 600\,p^8 + 450\,p^{10} + 150\,p^{11} + 292\,p^3\,x + 300\,p^4\,x - 797\,p^5\,x - 223\,p^6\,x - \\
&\quad 671\,p^7\,x - 193\,p^8\,x + 894\,p^9\,x + 398\,p^{10}\,x - 158\,p^2\,x^2 + 596\,p^3\,x^2 - 790\,p^4\,x^2 - 510\,p^5\,x^2 - \\
&\quad 341\,p^6\,x^2 - 677\,p^7\,x^2 + 1136\,p^8\,x^2 + 744\,p^9\,x^2 - 77\,p\,x^3 - 85\,p^2\,x^3 + 312\,p^3\,x^3 - 714\,p^4\,x^3 - \\
&\quad 219\,p^5\,x^3 - 657\,p^6\,x^3 + 659\,p^7\,x^3 + 781\,p^8\,x^3 - 154\,p\,x^4 + 503\,p^2\,x^4 - 281\,p^3\,x^4 - \\
&\quad 477\,p^4\,x^4 - 493\,p^5\,x^4 + 237\,p^6\,x^4 + 665\,p^7\,x^4 + 25\,p\,x^5 + 477\,p^2\,x^5 - 676\,p^3\,x^5 - 326\,p^4\,x^5 + \\
&\quad 122\,p^5\,x^5 + 378\,p^6\,x^5 + 204\,p\,x^6 - 198\,p^2\,x^6 - 396\,p^3\,x^6 + 131\,p^4\,x^6 + 259\,p^5\,x^6 + 77\,p\,x^7 - \\
&\quad 313\,p^2\,x^7 + 84\,p^3\,x^7 + 152\,p^4\,x^7 - 50\,p\,x^8 - 43\,p^2\,x^8 + 93\,p^3\,x^8 - 25\,p\,x^9 + 25\,p^2\,x^9)\,S_p + \\
&\ (-1200\,p - 4800\,p^2 - 5700\,p^3 + 2100\,p^4 + 10\,950\,p^5 + 9000\,p^6 - 150\,p^7 - 5100\,p^8 - \\
&\quad 3750\,p^9 - 1200\,p^{10} - 150\,p^{11} - 564\,x - 2058\,p\,x + 38\,p^2\,x + 10\,241\,p^3\,x + 18\,149\,p^4\,x + \\
&\quad 9606\,p^5\,x - 7554\,p^6\,x - 14\,703\,p^7\,x - 9671\,p^8\,x - 3086\,p^9\,x - 398\,p^{10}\,x - 306\,x^2 + \\
&\quad 2371\,p\,x^2 + 12\,540\,p^2\,x^2 + 19\,339\,p^3\,x^2 + 6116\,p^4\,x^2 - 17\,447\,p^5\,x^2 - 26\,290\,p^6\,x^2 - \\
&\quad 17\,019\,p^7\,x^2 - 5560\,p^8\,x^2 - 744\,p^9\,x^2 + 1350\,x^3 + 5097\,p\,x^3 + 4941\,p^2\,x^3 - 6553\,p^3\,x^3 - \\
&\quad 22\,131\,p^4\,x^3 - 26\,174\,p^5\,x^3 - 16\,598\,p^6\,x^3 - 5589\,p^7\,x^3 - 781\,p^8\,x^3 + 526\,x^4 - 673\,p\,x^4 - \\
&\quad 6996\,p^2\,x^4 - 15\,232\,p^3\,x^4 - 17\,732\,p^4\,x^4 - 12\,050\,p^5\,x^4 - 4418\,p^6\,x^4 - 665\,p^7\,x^4 - 1058\,x^5 - \\
&\quad 3311\,p\,x^5 - 4999\,p^2\,x^5 - 5712\,p^3\,x^5 - 4734\,p^4\,x^5 - 2146\,p^5\,x^5 - 378\,p^6\,x^5 - 134\,x^6 - \\
&\quad 183\,p\,x^6 - 814\,p^2\,x^6 - 1670\,p^3\,x^6 - 1164\,p^4\,x^6 - 259\,p^5\,x^6 + 322\,x^7 + 347\,p\,x^7 - 347\,p^2\,x^7 - \\
&\quad 524\,p^3\,x^7 - 152\,p^4\,x^7 - 86\,x^8 - 315\,p\,x^8 - 322\,p^2\,x^8 - 93\,p^3\,x^8 - 50\,x^9 - 75\,p\,x^9 - 25\,p^2\,x^9)\}
\end{aligned}
$$

In[∘]:=
```
(* Convert the second-order recurrence into an operator *)
annDp = {NormalizeCoefficients[
    ToOrePolynomial[f[p, x] == (DxH[n, x] /. DxH → f /. n → p)[[6, 1]], f[p, x]]]};
Support[
 annDp]
```

Out[∘]= $\{\{S_p^2,\ S_p,\ 1\}\}$

In[∘]:=
```
(* Prepare the bivariate summand for the second sum *)
annSmnd = DFiniteTimes[
   ToOrePolynomial[Append[annDp /. p → i, S[p] - 1], OreAlgebra[S[i], S[p]]],
   Annihilator[R3[i, p], {S[i], S[p]}]];
ByteCount[annSmnd]
```

Out[∘]= 223 648

```
ct = CreativeTelescoping[annSmnd, S[i] - 1]
```

Out[∘]=

$$
\begin{aligned}
&\{\{1\}, \\
&\ \{((2500\,i^3 + 3750\,i^4 - 1750\,i^5 - 4500\,i^6 + 2250\,i^8 - 750\,i^9 - 2250\,i^{10} - 250\,i^{11} + 750\,i^{12} + 250\,i^{13} - \\
&\qquad 3600\,i^3\,p - 5200\,i^4\,p + 2100\,i^5\,p + 4300\,i^6\,p + 1200\,i^8\,p + 2700\,i^9\,p + 100\,i^{10}\,p - \\
&\qquad 1200\,i^{11}\,p - 400\,i^{12}\,p - 1250\,i^2\,x - 10\,902\,i^3\,x - 12\,364\,i^4\,x + 5247\,i^5\,x + 9651\,i^6\,x - \\
&\qquad 516\,i^8\,x + 2139\,i^9\,x + 807\,i^{10}\,x - 234\,i^{11}\,x - 78\,i^{12}\,x + 1800\,i^2\,p\,x + 4200\,i^3\,p\,x + \\
&\qquad 1650\,i^4\,p\,x + 2630\,i^5\,p\,x + 5820\,i^6\,p\,x - 4260\,i^8\,p\,x - 1560\,i^9\,p\,x + 390\,i^{10}\,p\,x + \\
&\qquad 130\,i^{11}\,p\,x - 1250\,i\,x^2 + 3576\,i^2\,x^2 + 12\,555\,i^3\,x^2 + 9057\,i^4\,x^2 - 1671\,i^5\,x^2 - 1671\,i^6\,x^2 - \\
&\qquad 234\,i^8\,x^2 - 156\,i^9\,x^2 + 1800\,i\,p\,x^2 - 60\,i^2\,p\,x^2 + 2445\,i^3\,p\,x^2 + 3045\,i^4\,p\,x^2 - 3075\,i^5\,p\,x^2 - \\
&\qquad 4005\,i^6\,p\,x^2 + 390\,i^8\,p\,x^2 + 3576\,i\,x^3 + 541\,i^2\,x^3 + 7794\,i^3\,x^3 + 9601\,i^4\,x^3 - 4545\,i^5\,x^3 -
\end{aligned}
$$

$8403\,i^6\,x^3 + 750\,i^8\,x^3 - 1515\,i^9\,x^3 - 755\,i^{10}\,x^3 + 265\,i\,p\,x^3 - 2625\,i^2\,p\,x^3 - 7840\,i^3\,p\,x^3 -$
$3990\,i^4\,p\,x^3 - 1590\,i^5\,p\,x^3 - 4260\,i^6\,p\,x^3 + 3480\,i^8\,p\,x^3 + 1560\,i^9\,p\,x^3 + 1828\,i\,x^4 -$
$6606\,i^2\,x^4 - 23\,160\,i^3\,x^4 - 16\,632\,i^4\,x^4 + 3186\,i^5\,x^4 + 3186\,i^6\,x^4 + 234\,i^8\,x^4 + 156\,i^9\,x^4 -$
$15\,p\,x^4 - 4380\,i\,p\,x^4 + 640\,i^2\,p\,x^4 - 2160\,i^3\,p\,x^4 - 3360\,i^4\,p\,x^4 + 4590\,i^5\,p\,x^4 +$
$5670\,i^6\,p\,x^4 - 390\,i^8\,p\,x^4 - 10\,182\,i\,x^5 + 1893\,i^2\,x^5 + 5502\,i^3\,x^5 + 3453\,i^4\,x^5 +$
$156\,i^6\,x^5 - 405\,i\,p\,x^5 + 1695\,i^2\,p\,x^5 + 6240\,i^3\,p\,x^5 + 2340\,i^4\,p\,x^5 - 780\,i^5\,p\,x^5 +$
$1256\,i\,x^6 + 3030\,i^2\,x^6 + 10\,605\,i^3\,x^6 + 7575\,i^4\,x^6 - 1515\,i^5\,x^6 - 1515\,i^6\,x^6 + 60\,p\,x^6 +$
$4140\,i\,p\,x^6 - 840\,i^2\,p\,x^6 - 285\,i^3\,p\,x^6 + 315\,i^4\,p\,x^6 - 1515\,i^5\,p\,x^6 - 1665\,i^6\,p\,x^6 +$
$9636\,i\,x^7 - 1659\,i^2\,x^7 - 4956\,i^3\,x^7 - 3219\,i^4\,x^7 - 78\,i^6\,x^7 + 15\,i\,p\,x^7 - 915\,i^2\,p\,x^7 -$
$3900\,i^3\,p\,x^7 - 1170\,i^4\,p\,x^7 + 390\,i^5\,p\,x^7 - 3486\,i\,x^8 - 90\,p\,x^8 - 2340\,i\,p\,x^8 +$
$390\,i^2\,p\,x^8 - 3030\,i\,x^9 + 475\,i^2\,x^9 + 1470\,i^3\,x^9 + 995\,i^4\,x^9 + 125\,i\,p\,x^9 + 45\,i^2\,p\,x^9 +$
$520\,i^3\,p\,x^9 + 2142\,i\,x^{10} + 60\,p\,x^{10} + 780\,i\,p\,x^{10} - 130\,i^2\,p\,x^{10} - 490\,i\,x^{12} - 15\,p\,x^{12}) \,/$
$(60\,(1 + i)\,(1 + i - p)\,(-100\,i^3 - 450\,i^4 - 600\,i^5 + 600\,i^7 + 450\,i^8 + 100\,i^9 - 52\,i^2\,x +$
$199\,i^3\,x + 831\,i^4\,x + 805\,i^5\,x + 147\,i^6\,x - 104\,i^7\,x - 26\,i^8\,x + 101\,i\,x^2 + 277\,i^2\,x^2 +$
$98\,i^3\,x^2 - 130\,i^4\,x^2 - 52\,i^5\,x^2 - 3\,x^3 - 26\,i\,x^3 + 26\,i^2\,x^3 - 147\,i^3\,x^3 - 701\,i^4\,x^3 -$
$753\,i^5\,x^3 - 251\,i^6\,x^3 - 202\,i\,x^4 - 580\,i^2\,x^4 - 300\,i^3\,x^4 + 130\,i^4\,x^4 + 52\,i^5\,x^4 + 9\,x^5 +$
$52\,i\,x^5 + 52\,i^2\,x^5 + 101\,i\,x^6 + 303\,i^2\,x^6 + 202\,i^3\,x^6 - 9\,x^7 - 26\,i\,x^7 - 26\,i^2\,x^7 + 3\,x^9)))$
$S_i + (-2500\,i^2 - 6250\,i^3 - 500\,i^4 + 10\,000\,i^5 + 1500\,i^6 - 13\,500\,i^7 - 1500\,i^8 +$
$14\,250\,i^9 + 5500\,i^{10} - 4250\,i^{11} - 2500\,i^{12} - 250\,i^{13} + 3600\,i^2\,p +$
$7800\,i^3\,p + 1600\,i^4\,p - 900\,i^5\,p + 4700\,i^6\,p - 7200\,i^7\,p - 17\,400\,i^8\,p -$
$3300\,i^9\,p + 7100\,i^{10}\,p + 3600\,i^{11}\,p + 400\,i^{12}\,p + 1250\,i\,x + 12\,152\,i^2\,x +$
$24\,046\,i^3\,x + 2572\,i^4\,x - 20\,613\,i^5\,x + 219\,i^6\,x + 9996\,i^7\,x -$
$7728\,i^8\,x - 7881\,i^9\,x + 207\,i^{10}\,x + 702\,i^{11}\,x + 78\,i^{12}\,x - 1800\,i\,p\,x -$
$7020\,i^2\,p\,x - 4550\,i^3\,p\,x - 5980\,i^4\,p\,x - 21\,350\,i^5\,p\,x - 8880\,i^6\,p\,x +$
$17\,160\,i^7\,p\,x + 12\,120\,i^8\,p\,x - 130\,i^9\,p\,x - 1040\,i^{10}\,p\,x - 130\,i^{11}\,p\,x +$
$1250\,x^2 - 2326\,i\,x^2 - 17\,646\,i^2\,x^2 - 22\,737\,i^3\,x^2 - 2061\,i^4\,x^2 + 4077\,i^5\,x^2 -$
$2919\,i^6\,x^2 + 624\,i^7\,x^2 + 1170\,i^8\,x^2 + 156\,i^9\,x^2 - 1800\,p\,x^2 - 990\,i\,p\,x^2 -$
$1650\,i^2\,p\,x^2 - 6645\,i^3\,p\,x^2 + 5355\,i^4\,p\,x^2 + 13\,155\,i^5\,p\,x^2 + 2445\,i^6\,p\,x^2 -$
$1560\,i^7\,p\,x^2 - 390\,i^8\,p\,x^2 - 3576\,x^3 - 4072\,i\,x^3 - 8035\,i^2\,x^3 - 18\,520\,i^3\,x^3 -$
$1681\,i^4\,x^3 + 18\,663\,i^5\,x^3 + 873\,i^6\,x^3 - 9060\,i^7\,x^3 + 4530\,i^8\,x^3 + 6035\,i^9\,x^3 +$
$755\,i^{10}\,x^3 + 210\,p\,x^3 + 3575\,i\,p\,x^3 + 12\,435\,i^2\,p\,x^3 + 10\,660\,i^3\,p\,x^3 +$
$6240\,i^4\,p\,x^3 + 16\,410\,i^5\,p\,x^3 + 7320\,i^6\,p\,x^3 - 13\,260\,i^7\,p\,x^3 - 10\,560\,i^8\,p\,x^3 -$
$1560\,i^9\,p\,x^3 - 1828\,x^4 + 4778\,i\,x^4 + 32\,796\,i^2\,x^4 + 42\,432\,i^3\,x^4 + 3576\,i^4\,x^4 -$
$8622\,i^5\,x^4 + 4434\,i^6\,x^4 - 624\,i^7\,x^4 - 1170\,i^8\,x^4 - 156\,i^9\,x^4 + 4265\,p\,x^4 +$
$2110\,i\,p\,x^4 + 50\,i^2\,p\,x^4 + 8220\,i^3\,p\,x^4 - 10\,320\,i^4\,p\,x^4 - 21\,630\,i^5\,p\,x^4 -$
$4110\,i^6\,p\,x^4 + 1560\,i^7\,p\,x^4 + 390\,i^8\,p\,x^4 + 10\,182\,x^5 + 8154\,i\,x^5 -$
$7905\,i^2\,x^5 - 8310\,i^3\,x^5 - 2673\,i^4\,x^5 - 936\,i^5\,x^5 - 156\,i^6\,x^5 - 1020\,p\,x^5 -$
$5325\,i\,p\,x^5 - 10\,785\,i^2\,p\,x^5 - 7800\,i^3\,p\,x^5 + 780\,i^5\,p\,x^5 - 1256\,x^6 -$
$4286\,i\,x^6 - 15\,150\,i^2\,x^6 - 19\,695\,i^3\,x^6 - 1515\,i^4\,x^6 + 4545\,i^5\,x^6 - 1515\,i^6\,x^6 -$
$3810\,p\,x^6 - 2160\,i\,p\,x^6 + 1860\,i^2\,p\,x^6 - 1575\,i^3\,p\,x^6 + 4965\,i^4\,p\,x^6 +$
$8475\,i^5\,p\,x^6 + 1665\,i^6\,p\,x^6 - 9636\,x^7 - 7842\,i\,x^7 + 6735\,i^2\,x^7 + 7920\,i^3\,x^7 +$
$2829\,i^4\,x^7 + 468\,i^5\,x^7 + 78\,i^6\,x^7 + 1410\,p\,x^7 + 5325\,i\,p\,x^7 + 7665\,i^2\,p\,x^7 +$
$4680\,i^3\,p\,x^7 - 390\,i^5\,p\,x^7 + 3486\,x^8 + 3486\,i\,x^8 + 2040\,p\,x^8 + 1560\,i\,p\,x^8 -$

$$
\begin{aligned}
&390\, i^2\, p\, x^8 + 3030\, x^9 + 2510\, i\, x^9 - 1855\, i^2\, x^9 - 2510\, i^3\, x^9 - 995\, i^4\, x^9 - \\
&600\, p\, x^9 - 1775\, i\, p\, x^9 - 1515\, i^2\, p\, x^9 - 520\, i^3\, p\, x^9 - 2142\, x^{10} - 2142\, i\, x^{10} - \\
&710\, p\, x^{10} - 520\, i\, p\, x^{10} + 130\, i^2\, p\, x^{10} + 490\, x^{12} + 490\, i\, x^{12} + 15\, p\, x^{12}\big) \Big/ \\
&\big(60\, i\, (i - p)\, \big(-100\, i^3 - 450\, i^4 - 600\, i^5 + 600\, i^7 + 450\, i^8 + 100\, i^9 - 52\, i^2\, x + 199\, i^3\, x + \\
&\quad 831\, i^4\, x + 805\, i^5\, x + 147\, i^6\, x - 104\, i^7\, x - 26\, i^8\, x + 101\, i\, x^2 + 277\, i^2\, x^2 + 98\, i^3\, x^2 - \\
&\quad 130\, i^4\, x^2 - 52\, i^5\, x^2 - 3\, x^3 - 26\, i\, x^3 + 26\, i^2\, x^3 - 147\, i^3\, x^3 - 701\, i^4\, x^3 - 753\, i^5\, x^3 - \\
&\quad 251\, i^6\, x^3 - 202\, i\, x^4 - 580\, i^2\, x^4 - 300\, i^3\, x^4 + 130\, i^4\, x^4 + 52\, i^5\, x^4 + 9\, x^5 + 52\, i\, x^5 + \\
&\quad 52\, i^2\, x^5 + 101\, i\, x^6 + 303\, i^2\, x^6 + 202\, i^3\, x^6 - 9\, x^7 - 26\, i\, x^7 - 26\, i^2\, x^7 + 3\, x^9\big)\big)\}\}
\end{aligned}
$$

In[∘]:=
```
(* Telescoper is 1! Hence the following
  is an antidifference (i.e., an annihilator for it): *)
cert = DFiniteOreAction[annSmnd, ct[[2, 1]]];
ByteCount[cert]
```

Out[∘]= 295 600

In[∘]:=
```
(* Sanity check,
that the above gives indeed an antidifference for the summand. *)
test = smnd[i, p] + ApplyOreOperator[(S[i] - 1) ** ct[[2, 1]], smnd[i, p]];
Together[Table[test, {p, 10, 15}, {i, 5}] /. smnd[i_, p_] :> R3[i, p] * DxR[i, x]]
```

Out[∘]= {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}

In[∘]:=
```
(* Evaluate the antidifference at the upper summation bound: *)
annUpper = DFiniteSubstitute[cert, {i → p}]
```

Out[∘]=

$$
\begin{aligned}
&\{\big(3750\, p^2 + 15\,650\, p^3 + 16\,500\, p^4 - 19\,350\, p^5 - 61\,750\, p^6 - 60\,750\, p^7 - 27\,300\, p^8 + 600\, p^9 + 11\,400\, p^{10} + \\
&\quad 10\,000\, p^{11} + 4500\, p^{12} + 1050\, p^{13} + 100\, p^{14} + 1875\, p\, x - 5728\, p^2\, x - 65\,923\, p^3\, x - 176\,865\, p^4\, x - \\
&\quad 225\,026\, p^5\, x - 142\,039\, p^6\, x - 26\,320\, p^7\, x + 18\,995\, p^8\, x + 10\,628\, p^9\, x - 409\, p^{10}\, x - 1731\, p^{11}\, x - \\
&\quad 490\, p^{12}\, x - 39\, p^{13}\, x - 5989\, p\, x^2 - 14\,665\, p^2\, x^2 + 10\,965\, p^3\, x^2 + 76\,649\, p^4\, x^2 + 110\,436\, p^5\, x^2 + \\
&\quad 77\,017\, p^6\, x^2 + 28\,013\, p^7\, x^2 + 3657\, p^8\, x^2 - 1178\, p^9\, x^2 - 472\, p^{10}\, x^2 - 61\, p^{11}\, x^2 + 296\, p\, x^3 + \\
&\quad 10\,764\, p^2\, x^3 + 64\,210\, p^3\, x^3 + 154\,524\, p^4\, x^3 + 183\,949\, p^5\, x^3 + 105\,323\, p^6\, x^3 + 8619\, p^7\, x^3 - \\
&\quad 26\,496\, p^8\, x^3 - 17\,410\, p^9\, x^3 - 5056\, p^{10}\, x^3 - 605\, p^{11}\, x^3 + 11\,361\, p\, x^4 + 30\,950\, p^2\, x^4 + \\
&\quad 9250\, p^3\, x^4 - 70\,288\, p^4\, x^4 - 124\,624\, p^5\, x^4 - 95\,172\, p^6\, x^4 - 36\,072\, p^7\, x^4 - 4914\, p^8\, x^4 + \\
&\quad 856\, p^9\, x^4 + 195\, p^{10}\, x^4 - 3977\, p\, x^5 - 16\,224\, p^2\, x^5 - 23\,299\, p^3\, x^5 - 11\,988\, p^4\, x^5 + 4933\, p^5\, x^5 + \\
&\quad 8164\, p^6\, x^5 + 2644\, p^7\, x^5 + 349\, p^8\, x^5 - 6642\, p\, x^6 - 16\,627\, p^2\, x^6 - 13\,934\, p^3\, x^6 + 9385\, p^4\, x^6 + \\
&\quad 30\,669\, p^5\, x^6 + 25\,259\, p^6\, x^6 + 9439\, p^7\, x^6 + 1515\, p^8\, x^6 + 1607\, p\, x^7 + 9617\, p^2\, x^7 + 14\,866\, p^3\, x^7 + \\
&\quad 9621\, p^4\, x^7 + 2535\, p^5\, x^7 - 386\, p^6\, x^7 - 156\, p^7\, x^7 + 1606\, p\, x^8 + 275\, p^2\, x^8 - 2846\, p^3\, x^8 - \\
&\quad 2525\, p^4\, x^8 - 1010\, p^5\, x^8 - 46\, p\, x^9 - 2266\, p^2\, x^9 - 4960\, p^3\, x^9 - 3995\, p^4\, x^9 - 1255\, p^5\, x^9 - \\
&\quad 581\, p\, x^{10} - 336\, p^2\, x^{10} + 245\, p^3\, x^{10} + 245\, p\, x^{11} + 245\, p^2\, x^{11} + 245\, p\, x^{12} + 245\, p^2\, x^{12}\big)\, S_p^2 + \\
&\big(-30\,000\, p^2 - 86\,800\, p^3 - 33\,500\, p^4 + 172\,600\, p^5 + 314\,900\, p^6 + 235\,800\, p^7 + 42\,600\, p^8 - \\
&\quad 87\,000\, p^9 - 99\,000\, p^{10} - 53\,600\, p^{11} - 16\,600\, p^{12} - 2800\, p^{13} - 200\, p^{14} - 15\,000\, p\, x + \\
&\quad 59\,924\, p^2\, x + 389\,956\, p^3\, x + 744\,502\, p^4\, x + 678\,278\, p^5\, x + 249\,000\, p^6\, x - 76\,032\, p^7\, x - \\
&\quad 110\,112\, p^8\, x - 34\,548\, p^9\, x + 2706\, p^{10}\, x + 4302\, p^{11}\, x + 1014\, p^{12}\, x + 78\, p^{13}\, x + 40\,412\, p\, x^2 + \\
&\quad 115\,006\, p^2\, x^2 + 58\,272\, p^3\, x^2 - 152\,532\, p^4\, x^2 - 276\,936\, p^5\, x^2 - 201\,168\, p^6\, x^2 - 71\,508\, p^7\, x^2 - \\
&\quad 8646\, p^8\, x^2 + 1560\, p^9\, x^2 + 390\, p^{10}\, x^2 + 14\,088\, p\, x^3 - 76\,924\, p^2\, x^3 - 379\,124\, p^3\, x^3 -
\end{aligned}
$$

$$646630\,p^4 x^3 - 532250\,p^5 x^3 - 144246\,p^6 x^3 + 124782\,p^7 x^3 + 141000\,p^8 x^3 + 61380\,p^9 x^3 +$$
$$13310\,p^{10} x^3 + 1210\,p^{11} x^3 - 75616\,p\,x^4 - 210248\,p^2 x^4 - 115452\,p^3 x^4 + 224412\,p^4 x^4 +$$
$$405906\,p^5 x^4 + 281748\,p^6 x^4 + 95748\,p^7 x^4 + 11676\,p^8 x^4 - 1560\,p^9 x^4 - 390\,p^{10} x^4 +$$
$$1644\,p\,x^5 + 76038\,p^2 x^5 + 131148\,p^3 x^5 + 91470\,p^4 x^5 + 25470\,p^5 x^5 + 312\,p^6 x^5 - 624\,p^7 x^5 +$$
$$38872\,p\,x^6 + 97076\,p^2 x^6 + 57180\,p^3 x^6 - 71880\,p^4 x^6 - 128970\,p^5 x^6 - 80580\,p^6 x^6 -$$
$$24240\,p^7 x^6 - 3030\,p^8 x^6 - 552\,p\,x^7 - 61764\,p^2 x^7 - 100104\,p^3 x^7 - 64560\,p^4 x^7 -$$
$$16500\,p^5 x^7 - 156\,p^6 x^7 + 312\,p^7 x^7 - 6972\,p\,x^8 - 3486\,p^2 x^8 - 180\,p\,x^9 + 15830\,p^2 x^9 +$$
$$23020\,p^3 x^9 + 12550\,p^4 x^9 + 2510\,p^5 x^9 + 4284\,p\,x^{10} + 2142\,p^2 x^{10} - 980\,p\,x^{12} - 490\,p^2 x^{12}\Big)\,S_p +$$
$$\Big(6600 + 41600\,p + 77450\,p^2 - 33850\,p^3 - 338200\,p^4 - 530050\,p^5 - 314050\,p^6 + 122550\,p^7 +$$
$$375900\,p^8 + 336400\,p^9 + 178600\,p^{10} + 61600\,p^{11} + 13600\,p^{12} + 1750\,p^{13} + 100\,p^{14} + 32542\,x +$$
$$132047\,p\,x + 171478\,p^2 x + 36941\,p^3 x - 9455\,p^4 x + 255170\,p^5 x + 519975\,p^6 x + 461592\,p^7 x +$$
$$221329\,p^8 x + 54228\,p^9 x + 2455\,p^{10} x - 2139\,p^{11} x - 524\,p^{12} x - 39\,p^{13} x - 58210\,x^2 -$$
$$279571\,p\,x^2 - 488999\,p^2 x^2 - 285449\,p^3 x^2 + 227733\,p^4 x^2 + 501120\,p^5 x^2 + 383475\,p^6 x^2 +$$
$$165031\,p^7 x^2 + 43821\,p^8 x^2 + 7498\,p^9 x^2 + 870\,p^{10} x^2 + 61\,p^{11} x^2 - 48656\,x^3 - 145096\,p\,x^3 -$$
$$85340\,p^2 x^3 + 165306\,p^3 x^3 + 241186\,p^4 x^3 - 23363\,p^5 x^3 - 298565\,p^6 x^3 - 309765\,p^7 x^3 -$$
$$163284\,p^8 x^3 - 49390\,p^9 x^3 - 8254\,p^{10} x^3 - 605\,p^{11} x^3 + 95690\,x^4 + 401105\,p\,x^4 + 610234\,p^2 x^4 +$$
$$309514\,p^3 x^4 - 214056\,p^4 x^4 - 400670\,p^5 x^4 - 247344\,p^6 x^4 - 74556\,p^7 x^4 - 8622\,p^8 x^4 +$$
$$704\,p^9 x^4 + 195\,p^{10} x^4 + 2450\,x^5 - 57567\,p\,x^5 - 230558\,p^2 x^5 - 357117\,p^3 x^5 - 290042\,p^4 x^5 -$$
$$134511\,p^5 x^5 - 35976\,p^6 x^5 - 5280\,p^7 x^5 - 349\,p^8 x^5 - 48944\,x^6 - 189354\,p\,x^6 - 250825\,p^2 x^6 -$$
$$93446\,p^3 x^6 + 91875\,p^4 x^6 + 122073\,p^5 x^6 + 60453\,p^6 x^6 + 14801\,p^7 x^6 + 1515\,p^8 x^6 + 15042\,x^7 +$$
$$89955\,p\,x^7 + 174997\,p^2 x^7 + 158088\,p^3 x^7 + 72429\,p^4 x^7 + 15123\,p^5 x^7 + 542\,p^6 x^7 - 156\,p^7 x^7 +$$
$$6516\,x^8 + 24556\,p\,x^8 + 34521\,p^2 x^8 + 23046\,p^3 x^8 + 7575\,p^4 x^8 + 1010\,p^5 x^8 - 888\,x^9 -$$
$$13972\,p\,x^9 - 28984\,p^2 x^9 - 23200\,p^3 x^9 - 8555\,p^4 x^9 - 1255\,p^5 x^9 - 2142\,x^{10} - 3703\,p\,x^{10} -$$
$$1806\,p^2 x^{10} - 245\,p^3 x^{10} - 490\,x^{11} - 735\,p\,x^{11} - 245\,p^2 x^{11} + 490\,x^{12} + 735\,p\,x^{12} + 245\,p^2 x^{12}\Big)\Big\}$$

In[∘]:=
```
(* Evaluate the antidifference at the lower summation bound: *)
annLower = DFiniteSubstitute[cert, {i → 1}]
```

Out[∘]=
$$\Big\{\left(3p + 7p^2 + 5p^3 + p^4 + 2p\,x + 3p^2 x + p^3 x + p\,x^2 + p^2 x^2\right) S_p^2 +$$
$$\left(-8p^2 - 8p^3 - 2p^4 - 4p\,x - 6p^2 x - 2p^3 x - 4p\,x^2 - 2p^2 x^2\right) S_p +$$
$$\left(-2 - 3p + p^2 + 3p^3 + p^4 + 2p\,x + 3p^2 x + p^3 x + 2x^2 + 3p\,x^2 + p^2 x^2\right)\Big\}$$

In[∘]:=
```
annSum2 = DFinitePlus[annUpper, annLower];
Support[annSum2]
```

Out[∘]= $\left\{\left\{S_p^4, S_p^3, S_p^2, S_p, 1\right\}\right\}$

In[∘]:=
```
(* Sanity check: *)
test = ApplyOreOperator[annSum2[[1]], f[p]];
Together[Table[test, {p, 10}] /. f[p_] :→ Sum[R3[i, p] * DxR[i, x], {i, 1, p - 1}]]
```

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
(* Putting all parts together, into a single recurrence *)
annTotal = DFinitePlus[annR1, annSum1, annSum2, annDp];
Support[annTotal]
```

Out[∘]= $\{\{S_p^6, S_p^5, S_p^4, S_p^3, S_p^2, S_p, 1\}\}$

```
(* Look at the integer roots of the leading coefficient: no positive ones. *)
(* Thus, initial values for p = 1,...,
6 need to be checked (has already been done above). *)
Cases[p /. Solve[LeadingCoefficient[annTotal[[1]]] == 0, p], _Integer]
```

Out[∘]= {-5, -4, -3, -2, -1, 0}

Show that CxH satisfies the definition of CxR

In[73]:=
```
CxR[p_, x_] := Together[Sum[DxR[i, x], {i, 1, p - 1}]];
Table[CxR[n, x], {n, 2, 6}]
```

Out[74]=

$$\Big\{\frac{2\,(-588 + 115\,x + 262\,x^2 + 15\,x^3)}{15\,x\,(1 + x)},$$

$$\big(2\,(33876 + 1599\,x - 8557\,x^2 - 10076\,x^3 + 1372\,x^4 + 1325\,x^5 + 45\,x^6)\big) \big/ \big(15\,(-1 + x)\,x\,(1 + x)\,(3 + 2\,x + x^2)\big),$$

$$\big(4\,(-1948032 - 195456\,x + 287378\,x^2 + 596151\,x^3 - 31920\,x^4 - 57357\,x^5 - 27786\,x^6 + 3081\,x^7 + 1864\,x^8 + 45\,x^9)\big) \big/ \big(15\,(-2 + x)\,(-1 + x)\,x\,(1 + x)\,(3 + 2\,x + x^2)\,(8 + 3\,x + x^2)\big),$$

$$\big(4\,(25833600 + 3248640\,x - 2459142\,x^2 - 7613863\,x^3 + 110635\,x^4 + 532050\,x^5 + 453862\,x^6 - 31332\,x^7 - 24048\,x^8 - 6482\,x^9 + 608\,x^{10} + 267\,x^{11} + 5\,x^{12})\big) \big/ \big((-3 + x)\,(-2 + x)\,(-1 + x)\,x\,(1 + x)\,(3 + 2\,x + x^2)\,(8 + 3\,x + x^2)\,(15 + 4\,x + x^2)\big),$$

$$\big(2\,(-49032345600 - 6985751040\,x + 3225832992\,x^2 + 13952147760\,x^3 + 125446686\,x^4 - 729560375\,x^5 - 896885950\,x^6 + 39219725\,x^7 + 37341112\,x^8 + 17921670\,x^9 - 1238736\,x^{10} - 610330\,x^{11} - 105446\,x^{12} + 8545\,x^{13} + 2942\,x^{14} + 45\,x^{15})\big) \big/ \big(3\,(-4 + x)\,(-3 + x)\,(-2 + x)\,(-1 + x)\,x\,(1 + x)\,(3 + 2\,x + x^2)\,(8 + 3\,x + x^2)\,(15 + 4\,x + x^2)\,(24 + 5\,x + x^2)\big)\Big\}$$

In[79]:=
```
Clear[myK, CxH];
myK[n_, x_] := myK[n, k] =
   Which[
    n == 1, 2 / 15 * (15 * x ^ 3 + 262 * x ^ 2 + 115 * x - 588) / (x + 1) / x,
    n == 2,
    2 / 15 * (45 * x ^ 6 + 1325 * x ^ 5 + 1372 * x ^ 4 - 10 076 * x ^ 3 - 8557 * x ^ 2 + 1599 * x + 33 876) /
        (x ^ 2 + 2 * x + 3) / x / (x + 1) / (x - 1),
    True, Together[(500 * n ^ 13 - 456 * n ^ 12 * x - 196 * n ^ 11 * x ^ 2 - 1706 * n ^ 10 * x ^ 3 +
            1726 * n ^ 9 * x ^ 4 + 946 * n ^ 8 * x ^ 5 + 1402 * n ^ 7 * x ^ 6 - 2084 * n ^ 6 * x ^ 7 -
            1304 * n ^ 5 * x ^ 8 - 226 * n ^ 4 * x ^ 9 + 814 * n ^ 3 * x ^ 10 + 554 * n ^ 2 * x ^ 11 +
            30 * n * x ^ 12 - 3250 * n ^ 12 + 2736 * n ^ 11 * x + 1078 * n ^ 10 * x ^ 2 +
            8530 * n ^ 9 * x ^ 3 - 7767 * n ^ 8 * x ^ 4 - 3784 * n ^ 7 * x ^ 5 - 4907 * n ^ 6 * x ^ 6 +
```

```
        6252 * n^5 * x^7 + 3260 * n^4 * x^8 + 452 * n^3 * x^9 - 1221 * n^2 * x^10 -
        554 * n * x^11 - 15 * x^12 + 6000 * n^11 - 3996 * n^10 * x - 4480 * n^9 * x^2 -
        12 635 * n^8 * x^3 + 9184 * n^7 * x^4 + 10 758 * n^6 * x^5 + 7519 * n^5 * x^6 -
        6275 * n^4 * x^7 - 6810 * n^3 * x^8 - 4124 * n^2 * x^9 + 547 * n * x^10 +
        262 * x^11 + 2750 * n^10 - 5100 * n^9 * x + 12 075 * n^8 * x^2 -
        640 * n^7 * x^3 + 4102 * n^6 * x^4 - 19 030 * n^5 * x^5 - 6530 * n^4 * x^6 +
        2130 * n^3 * x^7 + 6955 * n^2 * x^8 + 3898 * n * x^9 - 70 * x^10 -
        19 000 * n^9 + 15 517 * n^8 * x - 13 586 * n^7 * x^2 + 10 746 * n^6 * x^3 -
        14 138 * n^5 * x^4 + 16 200 * n^4 * x^5 + 12 799 * n^3 * x^6 + 7957 * n^2 * x^7 -
        2701 * n * x^8 - 1374 * x^9 + 11 250 * n^8 - 1372 * n^7 * x - 1253 * n^6 * x^2 +
        5828 * n^5 * x^3 + 6967 * n^4 * x^4 - 5098 * n^3 * x^5 - 15 122 * n^2 * x^6 -
        7980 * n * x^7 + 300 * x^8 + 16 000 * n^7 - 17 938 * n^6 * x + 13 177 * n^5 * x^2 -
        16 440 * n^4 * x^3 - 9334 * n^3 * x^4 - 6474 * n^2 * x^5 + 5499 * n * x^6 +
        2550 * x^7 - 17 750 * n^6 + 7100 * n^5 * x - 7025 * n^4 * x^2 + 5360 * n^3 * x^3 +
        14 263 * n^2 * x^4 + 6482 * n * x^5 - 330 * x^6 - 1500 * n^5 + 7589 * n^4 * x +
        3227 * n^3 * x^2 + 2803 * n^2 * x^3 - 5233 * n * x^4 - 2026 * x^5 +
        7000 * n^4 - 3364 * n^3 * x - 4875 * n^2 * x^2 - 1846 * n * x^3 + 115 * x^4 -
        2000 * n^3 - 716 * n^2 * x + 1858 * n * x^2 + 588 * x^3) / (n - 1) / (n - 1 - x) /
    (n^2 + n * x + x^2 - 1) / (250 * n^9 - 228 * n^8 * x - 98 * n^7 * x^2 -
     603 * n^6 * x^3 + 635 * n^5 * x^4 + 375 * n^4 * x^5 +
     98 * n^3 * x^6 - 407 * n^2 * x^7 - 277 * n * x^8 - 15 * x^9 -
     2250 * n^8 + 1824 * n^7 * x + 686 * n^6 * x^2 + 3618 * n^5 * x^3 -
     3175 * n^4 * x^4 - 1500 * n^3 * x^5 - 294 * n^2 * x^6 + 814 * n * x^7 +
     277 * x^8 + 8250 * n^7 - 5521 * n^6 * x - 2693 * n^5 * x^2 -
     8410 * n^4 * x^3 + 5715 * n^3 * x^4 + 2885 * n^2 * x^5 + 946 * n * x^6 -
     362 * x^7 - 15 750 * n^6 + 7590 * n^5 * x + 6605 * n^4 * x^2 +
     9520 * n^3 * x^3 - 4445 * n^2 * x^4 - 2770 * n * x^5 - 750 * x^6 +
     16 500 * n^5 - 4057 * n^4 * x - 9145 * n^3 * x^2 - 5870 * n^2 * x^3 +
     797 * n * x^4 + 965 * x^5 - 9000 * n^4 - 324 * n^3 * x +
     6503 * n^2 * x^2 + 2348 * n * x^3 + 473 * x^4 + 2000 * n^3 +
     716 * n^2 * x - 1858 * n * x^2 - 588 * x^3) * myK[n - 1, x] -
  n * (n - x) * (250 * n^9 - 228 * n^8 * x - 98 * n^7 * x^2 - 603 * n^6 * x^3 +
     635 * n^5 * x^4 + 375 * n^4 * x^5 + 98 * n^3 * x^6 -
     407 * n^2 * x^7 - 277 * n * x^8 - 15 * x^9 - 750 * n^7 +
     863 * n^6 * x - 635 * n^5 * x^2 + 635 * n^4 * x^3 - 635 * n^3 * x^4 +
     635 * n^2 * x^5 + 652 * n * x^6 + 45 * x^7 + 750 * n^5 -
     1042 * n^4 * x + 635 * n^3 * x^2 - 635 * n^2 * x^3 - 473 * n * x^4 -
     45 * x^5 - 250 * n^3 + 407 * n^2 * x + 98 * n * x^2 + 15 * x^3) *
   (n^2 + n * x + x^2 - 2 * n - x) / (n - 1) / (n - 1 - x) / (n^2 + n * x + x^2 - 1) /
    (250 * n^9 - 228 * n^8 * x - 98 * n^7 * x^2 - 603 * n^6 * x^3 + 635 * n^5 * x^4 +
     375 * n^4 * x^5 + 98 * n^3 * x^6 - 407 * n^2 * x^7 - 277 * n * x^8 -
     15 * x^9 - 2250 * n^8 + 1824 * n^7 * x + 686 * n^6 * x^2 + 3618 * n^5 * x^3 -
     3175 * n^4 * x^4 - 1500 * n^3 * x^5 - 294 * n^2 * x^6 + 814 * n * x^7 +
```

```
            277 * x^8 + 8250 * n^7 - 5521 * n^6 * x - 2693 * n^5 * x^2 - 8410 * n^4 * x^3 +
            5715 * n^3 * x^4 + 2885 * n^2 * x^5 + 946 * n * x^6 - 362 * x^7 -
            15 750 * n^6 + 7590 * n^5 * x + 6605 * n^4 * x^2 + 9520 * n^3 * x^3 -
            4445 * n^2 * x^4 - 2770 * n * x^5 - 750 * x^6 + 16 500 * n^5 - 4057 * n^4 * x -
            9145 * n^3 * x^2 - 5870 * n^2 * x^3 + 797 * n * x^4 + 965 * x^5 -
            9000 * n^4 - 324 * n^3 * x + 6503 * n^2 * x^2 + 2348 * n * x^3 + 473 * x^4 +
            2000 * n^3 + 716 * n^2 * x - 1858 * n * x^2 - 588 * x^3) * myK[n - 2, x]]];
CxH[n_, x_] := myK[n - 1, x];
Table[CxH[n, x] - CxR[n, x], {n, 2, 6}]
```

Out[82]= {0, 0, 0, 0, 0}

```
In[83]:= (* Prepare the bivariate summand for the sum *)
annSmnd = ToOrePolynomial[Append[annDp /. p → i, S[p] - 1], OreAlgebra[S[i], S[p]]];
```

In[85]:=
```
ct = CreativeTelescoping[annSmnd, S[i] - 1, S[p]]
```

Out[85]=
```
{{1},
 {((-500 i^3 - 250 i^4 + 1500 i^7 + 750 i^8 - 1500 i^9 - 750 i^10 + 500 i^11 + 250 i^12 + 250 i^2 x - 5668 i^3 x -
       1520 i^4 x + 6325 i^5 x - 1042 i^7 x + 2476 i^8 x + 863 i^9 x - 706 i^10 x - 228 i^11 x +
       250 i x^2 + 2834 i^2 x^2 + 1309 i^3 x^2 + 3817 i^4 x^2 + 3126 i^5 x^2 - 1383 i^7 x^2 - 635 i^8 x^2 -
       218 i^9 x^2 - 98 i^10 x^2 + 3084 i x^3 - 1610 i^2 x^3 + 2099 i^3 x^3 + 2378 i^4 x^3 - 4104 i^5 x^3 +
       1385 i^7 x^3 - 880 i^8 x^3 - 853 i^9 x^3 + 1836 i x^4 - 7367 i^2 x^4 - 6776 i^3 x^4 - 2606 i^4 x^4 +
       684 i^5 x^4 + 1971 i^7 x^4 + 863 i^8 x^4 - 15 x^5 - 4859 i x^5 + 952 i^2 x^5 + 1391 i^3 x^5 +
       2444 i^4 x^5 + 1332 i^5 x^5 + 473 i^7 x^5 - 15 x^6 - 1225 i x^6 + 2636 i^2 x^6 + 3748 i^3 x^6 +
       1498 i^4 x^6 - 2934 i^5 x^6 + 45 x^7 + 2930 i x^7 - 3852 i^2 x^7 - 818 i^3 x^7 - 2706 i^4 x^7 -
       1743 i^5 x^7 + 45 x^8 - 2572 i x^8 - 697 i^2 x^8 + 473 i^3 x^8 - 1353 i^4 x^8 - 45 x^9 - 2133 i x^9 +
       2774 i^2 x^9 + 588 i^3 x^9 - 45 x^10 + 1711 i x^10 + 1108 i^2 x^10 + 15 x^11 + 978 i x^11 + 15 x^12) /
     (60 (1 + i) (-100 i^3 - 450 i^4 - 600 i^5 + 600 i^7 + 450 i^8 + 100 i^9 - 52 i^2 x + 199 i^3 x +
         831 i^4 x + 805 i^5 x + 147 i^6 x - 104 i^7 x - 26 i^8 x + 101 i x^2 + 277 i^2 x^2 + 98 i^3 x^2 -
         130 i^4 x^2 - 52 i^5 x^2 - 3 x^3 - 26 i x^3 + 26 i^2 x^3 - 147 i^3 x^3 - 701 i^4 x^3 - 753 i^5 x^3 -
         251 i^6 x^3 - 202 i x^4 - 580 i^2 x^4 - 300 i^3 x^4 + 130 i^4 x^4 + 52 i^5 x^4 + 9 x^5 + 52 i x^5 +
         52 i^2 x^5 + 101 i x^6 + 303 i^2 x^6 + 202 i^3 x^6 - 9 x^7 - 26 i x^7 - 26 i^2 x^7 + 3 x^9))) S_i +
   (500 i + 1250 i^2 + 2500 i^3 + 4000 i^4 - 4500 i^5 - 15 000 i^6 - 1500 i^7 + 14 250 i^8 +
     5500 i^9 - 4250 i^10 - 2500 i^11 - 250 i^12 + 250 x + 5918 i x + 14 136 i^2 x +
     88 i^3 x - 15 845 i^4 x + 87 i^5 x + 3838 i^6 x - 10 878 i^7 x - 5681 i^8 x +
     3057 i^9 x + 1802 i^10 x + 228 i^11 x + 2834 x^2 - 1044 i x^2 - 16 194 i^2 x^2 -
     19 484 i^3 x^2 - 15 822 i^4 x^2 - 6597 i^5 x^2 + 7247 i^6 x^2 + 7609 i^7 x^2 + 3083 i^8 x^2 +
     762 i^9 x^2 + 98 i^10 x^2 - 1723 x^3 - 12 979 i x^3 - 21 604 i^2 x^3 - 2930 i^3 x^3 +
     14 149 i^4 x^3 - 507 i^5 x^3 - 5779 i^6 x^3 + 7223 i^7 x^3 + 6797 i^8 x^3 + 853 i^9 x^3 -
     6595 x^4 + 3559 i x^4 + 26 542 i^2 x^4 + 14 517 i^3 x^4 + 4937 i^4 x^4 + 3631 i^5 x^4 -
     7247 i^6 x^4 - 4933 i^7 x^4 - 863 i^8 x^4 + 5044 x^5 + 11 983 i x^5 + 2302 i^2 x^5 -
     12 906 i^3 x^5 - 12 943 i^4 x^5 - 5521 i^5 x^5 - 1875 i^6 x^5 - 473 i^7 x^5 + 3870 x^6 -
     7515 i x^6 - 21 436 i^2 x^6 - 6872 i^3 x^6 + 1696 i^4 x^6 - 1498 i^5 x^6 - 6882 x^7 -
     8605 i x^7 + 4768 i^2 x^7 + 11 608 i^3 x^7 + 7445 i^4 x^7 + 1743 i^5 x^7 + 694 x^8 +
     6441 i x^8 + 8798 i^2 x^8 + 4449 i^3 x^8 + 1353 i^4 x^8 + 4274 x^9 + 4661 i x^9 - 426 i^2 x^9 -
     588 i^3 x^9 - 788 x^10 - 1941 i x^10 - 1108 i^2 x^10 - 963 x^11 - 978 i x^11 - 15 x^12) /
    (60 i (-100 i^3 - 450 i^4 - 600 i^5 + 600 i^7 + 450 i^8 + 100 i^9 - 52 i^2 x + 199 i^3 x +
       831 i^4 x + 805 i^5 x + 147 i^6 x - 104 i^7 x - 26 i^8 x + 101 i x^2 + 277 i^2 x^2 + 98 i^3 x^2 -
       130 i^4 x^2 - 52 i^5 x^2 - 3 x^3 - 26 i x^3 + 26 i^2 x^3 - 147 i^3 x^3 - 701 i^4 x^3 - 753 i^5 x^3 -
       251 i^6 x^3 - 202 i x^4 - 580 i^2 x^4 - 300 i^3 x^4 + 130 i^4 x^4 + 52 i^5 x^4 + 9 x^5 + 52 i x^5 +
       52 i^2 x^5 + 101 i x^6 + 303 i^2 x^6 + 202 i^3 x^6 - 9 x^7 - 26 i x^7 - 26 i^2 x^7 + 3 x^9))}}
```

In[86]:=
```
(* Telescoper is 1! Hence the following
   is an antidifference (i.e., an annihilator for it): *)
cert = DFiniteOreAction[annSmnd, ct[[2, 1]]];
ByteCount[cert]
```

Out[87]=
```
58 024
```

In[89]:=
```
Support[cert]
```

Out[89]= $\{\{S_p, 1\}, \{S_i^2, S_i, 1\}\}$

In[99]:=
```
(* Sanity check,
that the above gives indeed an antidifference for the summand. *)
test = smnd[i, p] + ApplyOreOperator[(S[i] - 1) ** ct[[2, 1]], smnd[i, p]];
Together[Table[test, {p, 8}, {i, p}] /. smnd[i_, p_] :→ DxR[i, x]]
```

Out[100]=
```
{{0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}
```

In[101]:=
```
(* Evaluate the antidifference at the upper summation bound: *)
annUpper = DFiniteSubstitute[cert, {i → p}]
```

Out[101]=

$$
\begin{aligned}
\{ (&-500 p^4 - 250 p^5 + 1500 p^8 + 750 p^9 - 1500 p^{10} - 750 p^{11} + 500 p^{12} + 250 p^{13} + 250 p^3 x - 5668 p^4 x - \\
&1520 p^5 x + 6325 p^6 x - 1042 p^8 x + 2476 p^9 x + 863 p^{10} x - 706 p^{11} x - 228 p^{12} x + 250 p^2 x^2 + \\
&2834 p^3 x^2 + 1309 p^4 x^2 + 3817 p^5 x^2 + 3126 p^6 x^2 - 1383 p^8 x^2 - 635 p^9 x^2 - 218 p^{10} x^2 - \\
&98 p^{11} x^2 + 3084 p^2 x^3 - 1610 p^3 x^3 + 2099 p^4 x^3 + 2378 p^5 x^3 - 4104 p^6 x^3 + 1385 p^8 x^3 - \\
&880 p^9 x^3 - 853 p^{10} x^3 + 1836 p^2 x^4 - 7367 p^3 x^4 - 6776 p^4 x^4 - 2606 p^5 x^4 + 684 p^6 x^4 + \\
&1971 p^8 x^4 + 863 p^9 x^4 - 15 p x^5 - 4859 p^2 x^5 + 952 p^3 x^5 + 1391 p^4 x^5 + 2444 p^5 x^5 + \\
&1332 p^6 x^5 + 473 p^8 x^5 - 15 p x^6 - 1225 p^2 x^6 + 2636 p^3 x^6 + 3748 p^4 x^6 + 1498 p^5 x^6 - \\
&2934 p^6 x^6 + 45 p x^7 + 2930 p^2 x^7 - 3852 p^3 x^7 - 818 p^4 x^7 - 2706 p^5 x^7 - 1743 p^6 x^7 + \\
&45 p x^8 - 2572 p^2 x^8 - 697 p^3 x^8 + 473 p^4 x^8 - 1353 p^5 x^8 - 45 p x^9 - 2133 p^2 x^9 + 2774 p^3 x^9 + \\
&588 p^4 x^9 - 45 p x^{10} + 1711 p^2 x^{10} + 1108 p^3 x^{10} + 15 p x^{11} + 978 p^2 x^{11} + 15 p x^{12}) S_p^2 + \\
(&500 p + 1750 p^2 + 3750 p^3 + 7000 p^4 - 250 p^5 - 19\,500 p^6 - 16\,500 p^7 + 11\,250 p^8 + 19\,000 p^9 + \\
&2750 p^{10} - 6000 p^{11} - 3250 p^{12} - 500 p^{13} + 250 x + 6168 p x + 20\,054 p^2 x + 13\,974 p^3 x - \\
&10\,089 p^4 x - 14\,238 p^5 x - 2400 p^6 x - 7040 p^7 x - 15\,517 p^8 x - 5100 p^9 x + 3996 p^{10} x + \\
&2736 p^{11} x + 456 p^{12} x + 2834 x^2 + 1790 p x^2 - 17\,488 p^2 x^2 - 38\,512 p^3 x^2 - 36\,615 p^4 x^2 - \\
&26\,236 p^5 x^2 - 2476 p^6 x^2 + 14\,856 p^7 x^2 + 12\,075 p^8 x^2 + 4480 p^9 x^2 + 1078 p^{10} x^2 + 196 p^{11} x^2 - \\
&1723 x^3 - 14\,702 p x^3 - 37\,667 p^2 x^3 - 22\,924 p^3 x^3 + 9120 p^4 x^3 + 11\,264 p^5 x^3 - 2182 p^6 x^3 + \\
&1444 p^7 x^3 + 12\,635 p^8 x^3 + 8530 p^9 x^3 + 1706 p^{10} x^3 - 6595 x^4 - 3036 p x^4 + 28\,265 p^2 x^4 + \\
&48\,426 p^3 x^4 + 26\,230 p^4 x^4 + 11\,174 p^5 x^4 - 4300 p^6 x^4 - 12\,180 p^7 x^4 - 7767 p^8 x^4 - 1726 p^9 x^4 + \\
&5044 x^5 + 17\,042 p x^5 + 19\,144 p^2 x^5 - 11\,556 p^3 x^5 - 27\,240 p^4 x^5 - 20\,908 p^5 x^5 - 8728 p^6 x^5 - \\
&2348 p^7 x^5 - 946 p^8 x^5 + 3870 x^6 - 3630 p x^6 - 27\,726 p^2 x^6 - 30\,944 p^3 x^6 - 8924 p^4 x^6 - \\
&1300 p^5 x^6 + 1436 p^6 x^6 - 6882 x^7 - 15\,532 p x^7 - 6767 p^2 x^7 + 20\,228 p^3 x^7 + 19\,871 p^4 x^7 + \\
&11\,894 p^5 x^7 + 3486 p^6 x^7 + 694 x^8 + 7090 p x^8 + 17\,811 p^2 x^8 + 13\,944 p^3 x^8 + 5329 p^4 x^8 + \\
&2706 p^5 x^8 + 4274 x^9 + 8980 p x^9 + 6368 p^2 x^9 - 3788 p^3 x^9 - 1176 p^4 x^9 - 788 x^{10} - 2684 p x^{10} - \\
&4760 p^2 x^{10} - 2216 p^3 x^{10} - 963 x^{11} - 1956 p x^{11} - 1956 p^2 x^{11} - 15 x^{12} - 30 p x^{12}) S_p + \\
(&-500 p - 1750 p^2 - 3750 p^3 - 12\,500 p^4 - 32\,500 p^5 - 43\,500 p^6 - 19\,500 p^7 + 23\,250 p^8 + 43\,250 p^9 + \\
&31\,750 p^{10} + 12\,750 p^{11} + 2750 p^{12} + 250 p^{13} - 250 x - 6168 p x - 20\,054 p^2 x - 17\,344 p^3 x + \\
&24\,577 p^4 x + 77\,558 p^5 x + 94\,235 p^6 x + 64\,160 p^7 x + 19\,139 p^8 x - 5176 p^9 x - 6419 p^{10} x - \\
&2030 p^{11} x - 228 p^{12} x - 2834 x^2 - 1790 p x^2 + 23\,298 p^2 x^2 + 58\,358 p^3 x^2 + 57\,806 p^4 x^2 + \\
&20\,499 p^5 x^2 - 11\,570 p^6 x^2 - 17\,976 p^7 x^2 - 10\,692 p^8 x^2 - 3845 p^9 x^2 - 860 p^{10} x^2 - 98 p^{11} x^2 + \\
&1723 x^3 + 14\,522 p x^3 + 32\,843 p^2 x^3 + 24\,534 p^3 x^3 - 18\,479 p^4 x^3 - 64\,522 p^5 x^3 - 80\,954 p^6 x^3 - \\
&61\,684 p^7 x^3 - 29\,080 p^8 x^3 - 7650 p^9 x^3 - 853 p^{10} x^3 + 6595 x^4 + 3036 p x^4 - 42\,221 p^2 x^4 - \\
&87\,979 p^3 x^4 - 72\,254 p^4 x^4 - 18\,768 p^5 x^4 + 14\,536 p^6 x^4 + 15\,300 p^7 x^4 + 5796 p^8 x^4 + \\
&863 p^9 x^4 - 5044 x^5 - 16\,487 p x^5 - 10\,625 p^2 x^5 + 16\,844 p^3 x^5 + 28\,969 p^4 x^5 + 18\,464 p^5 x^5 + \\
&7396 p^6 x^5 + 2348 p^7 x^5 + 473 p^8 x^5 - 3870 x^6 + 3645 p x^6 + 35\,011 p^2 x^6 + 52\,548 p^3 x^6 + \\
&35\,476 p^4 x^6 + 11\,922 p^5 x^6 + 1498 p^6 x^6 + 6882 x^7 + 14\,947 p x^7 + 1737 p^2 x^7 - 19\,496 p^3 x^7 - \\
&20\,613 p^4 x^7 - 9188 p^5 x^7 - 1743 p^6 x^7 - 694 x^8 - 7135 p x^8 - 15\,239 p^2 x^8 - 13\,247 p^3 x^8 - \\
&5802 p^4 x^8 - 1353 p^5 x^8 - 4274 x^9 - 8755 p x^9 - 4055 p^2 x^9 + 1014 p^3 x^9 + 588 p^4 x^9 + 788 x^{10} + \\
&2729 p x^{10} + 3049 p^2 x^{10} + 1108 p^3 x^{10} + 963 x^{11} + 1941 p x^{11} + 978 p^2 x^{11} + 15 x^{12} + 15 p x^{12}) \}
\end{aligned}
$$

In[102]:=
```
(* Evaluate the antidifference at the lower summation bound: *)
annLower = DFiniteSubstitute[cert, {i → 1}]
```

Out[102]= $\{S_p - 1\}$

In[103]:=
```
annSumC = DFinitePlus[annUpper, annLower];
Support[annSumC]
```

Out[104]= $\{\{S_p^3, S_p^2, S_p, 1\}\}$

In[108]:=
```
(* Sanity check: *)
test = ApplyOreOperator[annSumC[[1]], f[p]];
Together[Table[test, {p, 10}] /. f[p_] :→ Sum[DxR[i, x], {i, 1, p - 1}]]
```

Out[109]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[118]:=

```
(* Convert the second-order recurrence for K into an operator for C. *)
annC = {NormalizeCoefficients[
   ToOrePolynomial[(f[p, x] == (myK[n, x] /. myK → f /. n → p)[[6, 1]]) /.
     f[p_, x_] → f[p + 1, x], f[p, x]]]}
```

Out[118]=

$$\begin{aligned}
\{&(-500 p^{6} - 250 p^{7} + 1500 p^{8} + 750 p^{9} - 1500 p^{10} - 750 p^{11} + 500 p^{12} + 250 p^{13} + 1064 p^{5} x + 407 p^{6} x - \\
&2834 p^{7} x - 1042 p^{8} x + 2476 p^{9} x + 863 p^{10} x - 706 p^{11} x - 228 p^{12} x + 39 p^{4} x^{2} + 98 p^{5} x^{2} + \\
&1562 p^{6} x^{2} + 635 p^{7} x^{2} - 1383 p^{8} x^{2} - 635 p^{9} x^{2} - 218 p^{10} x^{2} - 98 p^{11} x^{2} - 475 p^{3} x^{3} + \\
&265 p^{4} x^{3} - 863 p^{5} x^{3} - 1385 p^{6} x^{3} + 1042 p^{7} x^{3} + 1385 p^{8} x^{3} - 880 p^{9} x^{3} - 853 p^{10} x^{3} - \\
&113 p^{2} x^{4} - 407 p^{3} x^{4} - 946 p^{4} x^{4} + 569 p^{5} x^{4} - 1270 p^{6} x^{4} - 1498 p^{7} x^{4} + 1971 p^{8} x^{4} + \\
&863 p^{9} x^{4} - 15 p x^{5} - 98 p^{2} x^{5} + 1018 p^{3} x^{5} - 680 p^{4} x^{5} + 1270 p^{5} x^{5} + 1270 p^{6} x^{5} + \\
&718 p^{7} x^{5} + 473 p^{8} x^{5} - 15 p x^{6} + 518 p^{2} x^{6} + 635 p^{3} x^{6} + 1304 p^{4} x^{6} + 17 p^{5} x^{6} - 814 p^{6} x^{6} + \\
&701 p^{7} x^{6} + 45 p x^{7} + 473 p^{2} x^{7} - 1197 p^{3} x^{7} + 680 p^{4} x^{7} - 1287 p^{5} x^{7} - 1042 p^{6} x^{7} + \\
&45 p x^{8} - 697 p^{2} x^{8} - 635 p^{3} x^{8} - 245 p^{4} x^{8} - 652 p^{5} x^{8} - 45 p x^{9} - 652 p^{2} x^{9} + 654 p^{3} x^{9} - \\
&113 p^{4} x^{9} - 45 p x^{10} + 292 p^{2} x^{10} + 407 p^{3} x^{10} + 15 p x^{11} + 277 p^{2} x^{11} + 15 p x^{12})\, S_p^{2} + \\
&(2000 p^{3} + 7000 p^{4} + 1500 p^{5} - 17\,750 p^{6} - 16\,000 p^{7} + 11\,250 p^{8} + 19\,000 p^{9} + 2750 p^{10} - \\
&6000 p^{11} - 3250 p^{12} - 500 p^{13} + 716 p^{2} x - 3364 p^{3} x - 7589 p^{4} x + 7100 p^{5} x + 17\,938 p^{6} x - \\
&1372 p^{7} x - 15\,517 p^{8} x - 5100 p^{9} x + 3996 p^{10} x + 2736 p^{11} x + 456 p^{12} x - 1858 p x^{2} - \\
&4875 p^{2} x^{2} - 3227 p^{3} x^{2} - 7025 p^{4} x^{2} - 13\,177 p^{5} x^{2} - 1253 p^{6} x^{2} + 13\,586 p^{7} x^{2} + \\
&12\,075 p^{8} x^{2} + 4480 p^{9} x^{2} + 1078 p^{10} x^{2} + 196 p^{11} x^{2} - 588 x^{3} - 1846 p x^{3} - 2803 p^{2} x^{3} + \\
&5360 p^{3} x^{3} + 16\,440 p^{4} x^{3} + 5828 p^{5} x^{3} - 10\,746 p^{6} x^{3} - 640 p^{7} x^{3} + 12\,635 p^{8} x^{3} + \\
&8530 p^{9} x^{3} + 1706 p^{10} x^{3} + 115 x^{4} + 5233 p x^{4} + 14\,263 p^{2} x^{4} + 9334 p^{3} x^{4} + 6967 p^{4} x^{4} + \\
&14\,138 p^{5} x^{4} + 4102 p^{6} x^{4} - 9184 p^{7} x^{4} - 7767 p^{8} x^{4} - 1726 p^{9} x^{4} + 2026 x^{5} + 6482 p x^{5} + \\
&6474 p^{2} x^{5} - 5098 p^{3} x^{5} - 16\,200 p^{4} x^{5} - 19\,030 p^{5} x^{5} - 10\,758 p^{6} x^{5} - 3784 p^{7} x^{5} - \\
&946 p^{8} x^{5} - 330 x^{6} - 5499 p x^{6} - 15\,122 p^{2} x^{6} - 12\,799 p^{3} x^{6} - 6530 p^{4} x^{6} - 7519 p^{5} x^{6} - \\
&4907 p^{6} x^{6} - 1402 p^{7} x^{6} - 2550 x^{7} - 7980 p x^{7} - 7957 p^{2} x^{7} + 2130 p^{3} x^{7} + 6275 p^{4} x^{7} + \\
&6252 p^{5} x^{7} + 2084 p^{6} x^{7} + 300 x^{8} + 2701 p x^{8} + 6955 p^{2} x^{8} + 6810 p^{3} x^{8} + 3260 p^{4} x^{8} + \\
&1304 p^{5} x^{8} + 1374 x^{9} + 3898 p x^{9} + 4124 p^{2} x^{9} + 452 p^{3} x^{9} + 226 p^{4} x^{9} - 70 x^{10} - 547 p x^{10} - \\
&1221 p^{2} x^{10} - 814 p^{3} x^{10} - 262 x^{11} - 554 p x^{11} - 554 p^{2} x^{11} - 15 x^{12} - 30 p x^{12})\, S_p + \\
&(-2000 p^{3} - 13\,000 p^{4} - 34\,500 p^{5} - 44\,750 p^{6} - 19\,750 p^{7} + 23\,250 p^{8} + 43\,250 p^{9} + \\
&31\,750 p^{10} + 12\,750 p^{11} + 2750 p^{12} + 250 p^{13} - 716 p^{2} x + 244 p^{3} x + 16\,409 p^{4} x + 53\,636 p^{5} x + \\
&79\,815 p^{6} x + 61\,326 p^{7} x + 19\,139 p^{8} x - 5176 p^{9} x - 6419 p^{10} x - 2030 p^{11} x - 228 p^{12} x + \\
&1858 p x^{2} + 10\,935 p^{2} x^{2} + 25\,907 p^{3} x^{2} + 29\,486 p^{4} x^{2} + 11\,159 p^{5} x^{2} - 11\,229 p^{6} x^{2} - \\
&17\,341 p^{7} x^{2} - 10\,692 p^{8} x^{2} - 3845 p^{9} x^{2} - 860 p^{10} x^{2} - 98 p^{11} x^{2} + 588 x^{3} + 1666 p x^{3} + \\
&1063 p^{2} x^{3} - 4885 p^{3} x^{3} - 23\,965 p^{4} x^{3} - 55\,845 p^{5} x^{3} - 75\,109 p^{6} x^{3} - 60\,642 p^{7} x^{3} - \\
&29\,080 p^{8} x^{3} - 7650 p^{9} x^{3} - 853 p^{10} x^{3} - 115 x^{4} - 5233 p x^{4} - 26\,270 p^{2} x^{4} - 55\,847 p^{3} x^{4} - \\
&58\,821 p^{4} x^{4} - 24\,907 p^{5} x^{4} + 8088 p^{6} x^{4} + 13\,802 p^{7} x^{4} + 5796 p^{8} x^{4} + 863 p^{9} x^{4} - 2026 x^{5} - \\
&5927 p x^{5} - 2716 p^{2} x^{5} + 10\,320 p^{3} x^{5} + 20\,000 p^{4} x^{5} + 17\,760 p^{5} x^{5} + 9488 p^{6} x^{5} + \\
&3066 p^{7} x^{5} + 473 p^{8} x^{5} + 330 x^{6} + 5514 p x^{6} + 20\,664 p^{2} x^{6} + 36\,404 p^{3} x^{6} + 35\,526 p^{4} x^{6} + \\
&19\,622 p^{5} x^{6} + 5721 p^{6} x^{6} + 701 p^{7} x^{6} + 2550 x^{7} + 7395 p x^{7} + 5384 p^{2} x^{7} - 4053 p^{3} x^{7} - \\
&8515 p^{4} x^{7} - 4965 p^{5} x^{7} - 1042 p^{6} x^{7} - 300 x^{8} - 2746 p x^{8} - 6258 p^{2} x^{8} - 6175 p^{3} x^{8} - \\
&3015 p^{4} x^{8} - 652 p^{5} x^{8} - 1374 x^{9} - 3673 p x^{9} - 3292 p^{2} x^{9} - 1106 p^{3} x^{9} - 113 p^{4} x^{9} + 70 x^{10} + \\
&592 p x^{10} + 929 p^{2} x^{10} + 407 p^{3} x^{10} + 262 x^{11} + 539 p x^{11} + 277 p^{2} x^{11} + 15 x^{12} + 15 p x^{12})\}
\end{aligned}$$

```
(* The third-order recurrence for the sum is a left-
 multiple of the guessed operator for C. *)
OreReduce[annSumC, annC]
```

Out[117]= {0}

```
In[119]:= (* Look at the integer roots of the leading coefficient: no positive ones. *)
(* Thus, initial values for p = 2,...,
4 need to be checked (has already been done above). *)
Cases[p /. Solve[LeadingCoefficient[annSumC[[1]]] == 0, p], _Integer]
```

Out[119]= {-1, 0}