Zeilberger's determinant problem (rough draft)

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9 January 2014

In [1] Zeilberger asks for a proof of a determinant evaluation: $\det M(d) = (-1)^d$. Here $M(d)$ is the $2d \times 2d$ matrix with entries $M_{2b-1,2b} = M_{2b,2b-1} = 1$, $M_{3b+1,2b-1} = M_{b-1,2b} = -1$ and all other entries zero.

By swapping columns $2b-1$ and $2b$ we get

$$\det M(d) = (-1)^d \det(I - N(d))$$

where $N(d)$'s only nonzero entries are $N_{b-1,2b-1} = N_{3b+1,2b} = 1$. We claim that either $N(d)$ is nilpotent (so that $\det(I - N(d)) = 1$ in line with Zeilberger's conjecture) or that 1 is an eigenvalue of $N(d)$ so that $\det(I - N(d)) = 0$.

Let e_j $(1 \le j \le 2d)$ be the column vector of height $2d$ with a 1 in the j-th row and zeros elsewhere. Let $e_j = 0$ if j is outside the range $1 \le j \le 2d$. Then $M(d)e_j = e_{\phi(j)}$ for $1 \le j \le 2d$ where

$$\phi(j) = \begin{cases} (j-1)/2 & \text{if } j \text{ is odd,} \\ 3j/2 + 1 & \text{if } j \text{ is even.} \end{cases}$$

If, starting with j repeatedly applying ϕ leads to a number outside $\{1, \dots, 2d\}$ then $M(d)^k e_j = 0$ for large enough k. If true for all $j \in \{1, \dots, 2d\}$ then $N(d)$ is nilpotent and so $\det M(d) = (-1)^d$. If however this iteration leads to a cycle $c_0, c_1, \dots, c_{t-1}, c_t = c_0$ with each $c_i \in \{1, \dots, 2d\}$ and $c_{i+1} = \phi(c_i)$ then $\sum_{i=0}^{t-1} e_{c_t}$ is a nonzero eigenvector of $N(d)$ with eigenvalue 1 and so $\det M(d) = 0$.

Consider the dynamics of ϕ on the set of positive integers. If there is some cycle of ϕ then for large enough d, $\det M(d) = 0$. Otherwise all trajectories hit 0 or are unbounded above, and then $\det M(d) = (-1)^d$.

Conjugating ϕ by the map $j \mapsto j+1$ gives a more familiar mapping. Define $\psi(j) = \phi(j-1) + 1$. Then

$$\psi(j) = \begin{cases} j/2 & \text{if } j \text{ is even,} \\ (3j+1)/2 & \text{if } j \text{ is even.} \end{cases}$$

which is equivalent to the familiar Collatz iteration. We conclude: if there is a nontrivial cycle in the Collatz problem, then $\det M(d) = 0$ for all large d, but if there are none then $\det M(d) = (-1)^d$ for all d.

References

[1] Doron Zeilberger, 'A conjectured explicit determinant evaluation whose proof would make me happy (and the OEIS richer)', preprint `arXiv:1401.1532`, 2104.